EE16B - Spring'20 - Lecture 6B Notes[1]

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State Space Models Continued

In the last lecture we considered a general state model of the form

$$\begin{aligned}
\frac{d}{dt}x_1(t) &= f_1(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)) \\
\frac{d}{dt}x_2(t) &= f_2(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)) \\
&\vdots \\
\frac{d}{dt}x_n(t) &= f_n(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)),
\end{aligned} \tag{1}$$

and rewrote it compactly as

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t), \vec{u}(t)). \tag{2}$$

When the system (2) is linear we write it in the matrix-vector form

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t), \tag{3}$$

where A is a $n \times n$ matrix and B is a $n \times m$ matrix.

Equilibrium States

Recall that, for a system without inputs, $\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t))$, the solutions of the static equation

$$f(\vec{x}) = 0$$

are called *equilibrium points*. We can extend the definition of an equilibrium to systems with inputs, assuming that a constant input $\vec{u}$ is applied instead of a time-varying one. In this case $\vec{x}$ is an equilibrium point if it satisfies

$$f(\vec{x}, \vec{u}) = 0, \tag{4}$$

where the solution depends on the constant input $\vec{u}$ applied.

For linear systems (3) we find equilibrium points by solving for $\vec{x}$ in

$$A\vec{x} + B\vec{u} = 0, \tag{5}$$

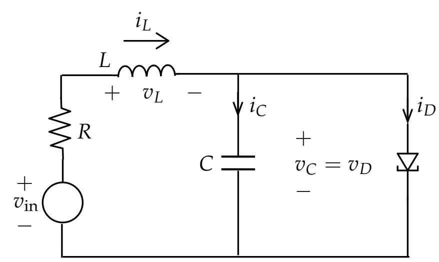
with $\vec{u}$ as a given constant. If A is invertible each constant input $\vec{u}$ produces a unique equilibrium.

When A is singular, there may be a continuum of infinitely many equilibrium points (*e.g.*, with $\vec{u} = 0$, every point in the null space of A is an equilibrium point) or there may be no equilibrium points, which happens when $B\vec{u}$ is not in the range space of A.

Note that multiple isolated equilibrium points – such as those in the pendulum example – can't occur in linear systems, since (5) has either a single solution, no solution, or a continuum of infinitely many solutions. Therefore, multiple isolated equilibria can arise only in nonlinear systems.

Example 1: In the last lecture we discussed the circuit on the right and obtained the state model:

$$\begin{aligned} \frac{d}{dt}v_C(t) &= \frac{1}{C}i_L(t) - \frac{1}{C}g(v_C(t)) \\ \frac{d}{dt}i_L(t) &= \frac{1}{L}\left(-v_C(t) - Ri_L(t) + v_{\text{in}}(t)\right), \end{aligned} \tag{6}$$



where g is a nonlinear function representing the tunnel diode's voltage-current characteristics (see figure below).

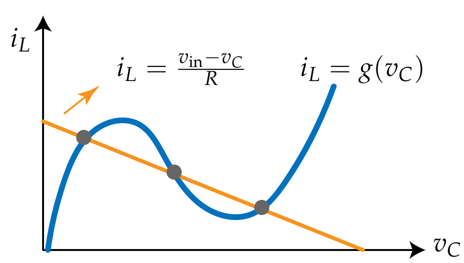
To find the equilibrium points we set the left-hand side of (6) to zero and solve for v_C and i_L:

$$\begin{aligned} \frac{1}{C}i_L - \frac{1}{C}g(v_C) &= 0 \\ \frac{1}{L}\left(-v_C - Ri_L + v_{\text{in}}\right) &= 0. \end{aligned}$$

It follows from these two equations that we can find the equilibrium points by superimposing the curves

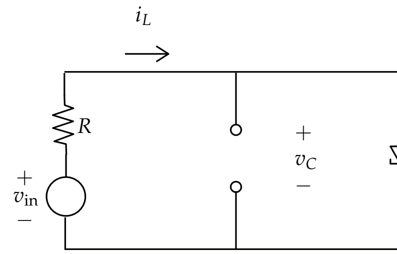
$$i_L = g(v_C) \quad \text{and} \quad i_L = \frac{v_{\text{in}} - v_C}{R}, \tag{7}$$

and finding their intersections. The figure below shows the case where there are three equilibrium points.



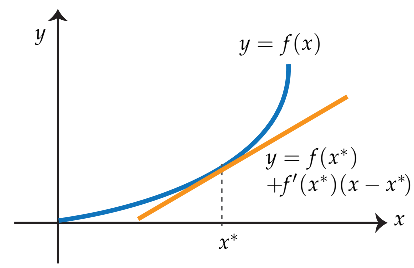
Depending on the values of the constants v_{in} and R, it is also possible to have only one or only two equilibrium points: imagine raising the orange line (*i.e.*, increasing v_{in}) until the two intersections on the left collapse into one, and then disappear.

To gain further insight into equilibrium states of circuits, recall that we use capacitor voltages and inductor currents as state variables. Thus, to find equilibrium points, we must solve the circuit equations with the time derivatives of capacitor voltages and inductor currents set to zero. Since $C\frac{d}{dt}v_C(t) = i_C(t)$ and $L\frac{d}{dt}i_L(t) = v_L(t)$, this means setting the capacitor currents and inductor voltages to zero. Thus, at equilibrium the capacitor acts like an open circuit and the inductor like a short circuit.



The figure on the right shows the tunnel diode circuit, with the inductor treated as short circuit and capacitor as open circuit. As an exercise show that i_L and v_C in this circuit indeed satisfy the equilibrium equations (7).

Linearization



Linear models are advantageous because their solutions can be found analytically. The methods applicable to nonlinear models are limited; therefore it is common practice to approximate a nonlinear model with a linear one that is valid around an equilibrium state.

Recall that the Taylor approximation of a differentiable function f around a point x^* is:

$$f(x) \approx f(x^*) + \nabla f(x)|_{x=x^*}(x - x^*),$$

as illustrated on the right for a scalar-valued function of a single variable. When $f(\vec{x})$ is a vector of n functions $f_1, \cdots, f_n$ as in our state models, $\nabla f(\vec{x})$ is interpreted as the $n \times n$ matrix of partial derivatives:

$$\nabla f(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial f_1(x_1,\ldots,x_n)}{\partial x_1} & \frac{\partial f_1(x_1,\ldots,x_n)}{\partial x_2} & \cdots & \frac{\partial f_1(x_1,\ldots,x_n)}{\partial x_n} \\ \frac{\partial f_2(x_1,\ldots,x_n)}{\partial x_1} & \frac{\partial f_2(x_1,\ldots,x_n)}{\partial x_2} & \cdots & \frac{\partial f_2(x_1,\ldots,x_n)}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n(x_1,\ldots,x_n)}{\partial x_1} & \frac{\partial f_n(x_1,\ldots,x_n)}{\partial x_2} & \cdots & \frac{\partial f_n(x_1,\ldots,x_n)}{\partial x_n} \end{bmatrix}.$$

We linearize nonlinear state models by applying this approximation around an equilibrium state. Let $\vec{x}^*$ be an equilbrium for the system

$$\frac{d}{dt}\vec{x}(t) = f(\vec{x}(t)), \tag{8}$$

that is $f(\vec{x}^*) = 0$, and define the deviation of $\vec{x}(t)$ from $\vec{x}^*$ as:

$$\boxed{\tilde{x}(t) := \vec{x}(t) - \vec{x}^*.} \tag{9}$$

Then we see that

$$\begin{aligned} \frac{d}{dt}\tilde{x}(t) &= \frac{d}{dt}\vec{x}(t) - \frac{d}{dt}\vec{x}^* \\ &= \frac{d}{dt}\vec{x}(t) = f(\vec{x}(t)) = f(\vec{x}^* + \tilde{x}(t)) \\ &\approx f(\vec{x}^*) + \nabla f(\vec{x})|_{\vec{x}=\vec{x}^*}\, \tilde{x}(t), \end{aligned} \tag{10}$$

where the second equality follows because $\vec{x}^*$ is constant and, thus, its derivative is zero. Substituting $f(\vec{x}^*) = 0$ in (10) and defining

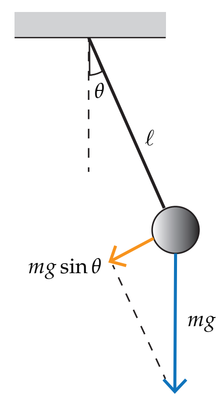
$$\boxed{A \triangleq \nabla f(\vec{x})|_{\vec{x}=\vec{x}^*}} \tag{11}$$

we obtain the linearization of (8) around the equilibrium $\vec{x}^*$ as:

$$\boxed{\frac{d}{dt}\tilde{x}(t) \approx A\tilde{x}(t).}$$

Example 2: Recall the pendulum model from the previous lecture:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_2(t) \\ \frac{dx_2(t)}{dt} &= -\frac{k}{m}x_2(t) - \frac{g}{\ell}\sin x_1(t) \end{aligned} \tag{12}$$



where

$$x_1(t) := \theta(t) \quad \text{and} \quad x_2(t) := \frac{d\theta(t)}{dt}.$$

The two distinct equilibrium points are the downward position:

$$x_1 = 0, \quad x_2 = 0, \tag{13}$$

and the upright position:

$$x_1 = \pi, \quad x_2 = 0. \tag{14}$$

Since the entries of $f(\vec{x})$ are $f_1(\vec{x}) = x_2$ and $f_2(\vec{x}) = -\frac{k}{m}x_2 - \frac{g}{\ell}\sin x_1$, we have

$$\nabla f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1(x_1,x_2)}{\partial x_1} & \frac{\partial f_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial f_2(x_1,x_2)}{\partial x_1} & \frac{\partial f_2(x_1,x_2)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x_1 & \frac{-k}{m} \end{bmatrix}.$$

By evaluating this matrix at (13) and (14), we obtain the linearization around the respective equilibrium point:

$$A_{\text{down}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & \frac{-k}{m} \end{bmatrix} \qquad A_{\text{up}} = \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & \frac{-k}{m} \end{bmatrix}. \tag{15}$$

As an exercise show that A_{up} has an eigenvalue with positive real part. We will see later that the presence of an eigenvalue with positive real part implies instability of the respective equilibrium state. In contrast A_{down} has eigenvalues with negative real parts, indicating stability of the downward position.