EE16B - Spring'20 - Lecture 13B Notes[1]

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Eigenvalue Assignment with State Feedback

In the previous lecture we studied the system

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t), \quad \vec{x}(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}, \tag{1}$$

with the feedback control policy

$$u(t) = k_1x_1(t) + k_2x_2(t) + \cdots + k_nx_n(t), \tag{2}$$

which we rewrote as

$$u(t) = K\vec{x}(t) \tag{3}$$

with $K = [k_1\, k_2\, \cdots\, k_n]$. When we substitute (3) in (1), we get

$$\vec{x}(t+1) = (A+BK)\vec{x}(t) \tag{4}$$

and the task is to choose K such that all eigenvalues of $A+BK$ are inside the unit circle[2] for stability.

Ideally we would like to be able to assign the eigenvalues as we wish, so we can influence the transients, *e.g.*, for faster convergence[3]. We claimed last time that the controllability of the system (1) gives us this ability: given a set of desired eigenvalues $\lambda_1, \cdots, \lambda_n$ we can find a corresponding K such that $A+BK$ has those eigenvalues.

[2] For continuous-time systems the eigenvalues of $A+BK$ must have negative real parts for stability.

[3] See Figures 1-2 in Lecture 12B to see how eigenvalue locations affect the transients.

In this lecture we review some examples of designing K. We then outline a proof of the claim that controllability gives us the ability to assign the eigenvalues arbitrarily.

Example 1 (Cruise Control): In Lecture 7A we studied the nonlinear model of a vehicle moving in a lane

$$M\frac{d}{dt}v(t) = -\frac{1}{2}\rho ac\, v(t)^2 + \frac{1}{R}u(t) \tag{5}$$

where $v(t)$ is velocity, $u(t)$ is the wheel torque, M is vehicle mass, ρ is air density, a is vehicle area, c is drag coefficient, and R is wheel radius. To maintain $v(t)$ at a desired value v^* we apply the torque

$$u^* = \frac{R}{2}\rho ac\, {v^*}^2,$$

which counterbalances the drag force at that velocity. We rewrite the model (5) as $\frac{d}{dt}v(t) = f(v(t), u(t))$, where

$$f(v,u) = -\frac{1}{2M}\rho ac\, v^2 + \frac{1}{RM}u.$$

Then the linearized dynamics for the perturbation $\tilde{v}(t) = v(t) - v^*$ is

$$\frac{d}{dt}\tilde{v}(t) = \lambda\tilde{v}(t) + b\tilde{u}(t), \tag{6}$$

where $\tilde{u}(t) = u(t) - u^*$,

$$\lambda = \left.\frac{\partial f(v,u)}{\partial v}\right|_{v^*,u^*} = -\frac{1}{M}\rho acv^*, \quad b = \left.\frac{\partial f(v,u)}{\partial u}\right|_{v^*,u^*} = \frac{1}{RM}.$$

If we apply $u(t) = u^*$, that is $\tilde{u}(t) = 0$, then the solution of (6) is

$$\tilde{v}(t) = \tilde{v}(0)e^{\lambda t},$$

which converges to 0 since $\lambda < 0$. This means that if $v(t)$ is perturbed from v^*, it will converge back to v^*. However, the rate of convergence can be very slow. Taking $M = 1700$ kg, $a = 2.6$ m^2, $\rho = 1.2$ kg/m^3, $c = 0.2$, which are reasonable for a sedan, and assuming $v^* = 29$ m/s ($\approx$ 65 mph) we get $\lambda \approx -0.01$ s^{-1}, *i.e.* a time constant of 100 seconds.

For faster convergence we can apply the feedback

$$\tilde{u}(t) = k\tilde{v}(t) \tag{7}$$

which leads to

$$\frac{d}{dt}\tilde{v}(t) = (\lambda + bk)\tilde{v}(t). \tag{8}$$

Then the convergence rate is determined by $\lambda + bk$, which we can assign arbitrarily by selecting k. Since $\tilde{u}(t) = u(t) - u^*$ and $\tilde{v}(t) = v(t) - v^*$, the actual torque applied to the vehicle is

$$u(t) = u^* + k(v(t) - v^*).$$

Example 2 (Robot Car): The robot car used in the lab has two wheels, each driven with a separate electric motor. Let $d_l(t)$ and $d_r(t)$ be the distance traveled by the left and right wheels, and let $u_l(t)$ and $u_r(t)$ denote the respective control inputs (duty cycle of pulse width modulated current). An appropriate model relating these variables is

$$\begin{aligned} d_l(t+1) - d_l(t) &= \theta_l u_l(t) - \beta_l \\ d_r(t+1) - d_r(t) &= \theta_r u_r(t) - \beta_r \end{aligned} \tag{9}$$

where the right hand sides approximate the speed for each wheel.

The parameters for the two wheels may be significantly different. Thus, applying an identical input to both wheels would lead to non-identical speeds, and the car would go in circles. To straighten the trajectory of the car we apply the control inputs

$$\begin{aligned} u_l(t) &= \frac{v^* + \beta_l}{\theta_l} + \frac{k_l}{\theta_l}(d_l(t) - d_r(t)) \\ u_r(t) &= \frac{v^* + \beta_r}{\theta_r} + \frac{k_r}{\theta_r}(d_l(t) - d_r(t)) \end{aligned} \tag{10}$$

where v^* is the desired velocity, and k_l and k_r are constants to be designed. Substitute (10) in (9) to get

$$\begin{aligned} d_l(t+1) - d_l(t) &= v^* + k_l(d_l(t) - d_r(t)) \\ d_r(t+1) - d_r(t) &= v^* + k_r(d_l(t) - d_r(t)). \end{aligned} \tag{11}$$

Next, define $\delta(t) := d_l(t) - d_r(t)$ and note from (11) that it satisfies

$$\delta(t+1) = (1 + k_l - k_r)\delta(t).$$

Thus, to ensure $\delta(t) \to 0$, we need to select k_l and k_r such that

$$|1 + k_l - k_r| < 1.$$

Without the feedback terms in (10), that is $k_l = k_r = 0$, we get

$$\delta(t+1) = \delta(t)$$

which means that the error accumulated in $\delta(t)$ persists and is in fact likely to grow if we incorporate a disturbance term. The feedback in (10) is thus essential to dissipate the error $\delta(t)$ and to keep it bounded in the presence of disturbances.

Example 3: Recall this example from the last lecture:

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t),$$

where the characteristic polynomial of A is

$$\det(\lambda I - A) = \lambda^2 - a_2\lambda - a_1.$$

If we substitute the control

$$u(t) = K\vec{x}(t) = k_1 x_1(t) + k_2 x_2(t)$$

the closed-loop system becomes

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ a_1 + k_1 & a_2 + k_2 \end{bmatrix}}_{A+BK} \vec{x}(t)$$

and, since $A + BK$ has the same structure as A with a_1, a_2 replaced by $a_1 + k_1$, $a_2 + k_2$, the eigenvalues of $A + BK$ are the roots of

$$\lambda^2 - (a_2 + k_2)\lambda - (a_1 + k_1).$$

If we want to assign the eigenvalues of $A + BK$ to desired values λ_1 and λ_2, we must match the polynomial above to

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.$$

This is accomplished with the choice

$$k_1 = -a_1 - \lambda_1\lambda_2, \quad k_2 = -a_2 + \lambda_1 + \lambda_2.$$

For example, if we want $\lambda_1 = \lambda_2 = 0$, then $k_1 = -a_1$ and $k_2 = -a_2$.

Example 4: Here is a three-state example where A and B have a structure similar to Example 2:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the characteristic polynomial is now

$$\det(\lambda I - A) = \lambda^3 - a_3\lambda^2 - a_2\lambda - a_1.$$

The closed-loop system is

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 + k_1 & a_2 + k_2 & a_3 + k_3 \end{bmatrix}}_{A+BK} \vec{x}(t)$$

which has characteristic polynomial

$$\det(\lambda I - (A + BK)) = \lambda^3 - (a_3 + k_3)\lambda^2 - (a_2 + k_2)\lambda - (a_1 + k_1). \quad (12)$$

Note that each one of k_1, k_2 and k_3 appears in precisely one coefficient and can change it to any desired value. If we want eigenvalues at λ_1, λ_2, λ_3, we simply match the coefficients of (12) to those of

$$\begin{aligned}(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \quad = \quad & \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_2)\lambda^2 \\ & -(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3\end{aligned}$$

by choosing $k_1 = \lambda_1\lambda_2\lambda_3 - a_1$, $k_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 - a_2$, and $k_3 = \lambda_1 + \lambda_2 + \lambda_3 - a_3$.

Why does controllability enable us to assign the eigenvalues?

We will now show that controllability allows us to arbitrarily assign the eigenvalues of $A + BK$ with the choice of K. The key to our argument is the special form of A and B in Examples 3 and 4, which we generalize to an arbitrary dimension n as:

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \qquad (13)$$

This structure is called the "controller canonical form," hence the subscript "c." When A_c has this form, the entries of the last row $a_1, \ldots, a_n$ appear as the coefficients of the characteristic polynomial:

$$\det(\lambda I - A_c) = \lambda^n - a_n\lambda^{n-1} - a_{n-1}\lambda^{n-2} - \cdots - a_2\lambda - a_1.$$

In addition $A_c + B_cK$ preserves the structure of A_c, except that the entry a_i is replaced by $a_i + k_i$, $i = 1, \ldots, n$. Therefore,

$$\det(\lambda I - (A_c + B_cK)) = \lambda^n - (a_n + k_n)\lambda^{n-1} \cdots - (a_2 + k_2)\lambda - (a_1 + k_1)$$

where each one of $k_1, \cdots, k_n$ appears in precisely one coefficient and can change it to any desired value. Thus we can arbitrarily assign the eigenvalues of $A_c + B_cK$ as we did in Examples 3 and 4.

So how do we prove that for any controllable system

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t) \tag{14}$$

we can assign the eigenvalues of $A + BK$ arbitrarily? We simply show that an appropriate change of variables $\vec{z} = T\vec{x}$ brings A and B to the form (13); that is, there exists T such that

$$TAT^{-1} = A_c \quad \text{and} \quad TB = B_c. \tag{15}$$

This means that we can design a state feedback $u = K_c\vec{z}$ to assign the eigenvalues of $A_c + B_cK_c$ as discussed above for the controller canonical form. Since $\vec{z} = T\vec{x}$, $u = K_c\vec{z}$ is identical to $u = K\vec{x}$ where

$$K = K_cT. \tag{16}$$

Note that $T(A + BK)T^{-1} = A_c + B_cK_c$ and, thus, the eigenvalues of $A + BK$ are identical[4] to those of $A_c + B_cK_c$, which have been assigned to desired values.

[4] If λ, $\vec{v}$ is an an eigenvalue/eigenvector pair for $A + BK$, that is

$$(A + BK)\vec{v} = \lambda\vec{v},$$

then λ is also an eigenvalue for $A_c + B_cK_c$, with eigenvector $T\vec{v}$. This is because

$$\begin{aligned}(A_c + B_cK_c)T\vec{v} &= (T(A + BK)T^{-1})T\vec{v} \\ &= T(A + BK)\vec{v} = T\lambda\vec{v} = \lambda T\vec{v}.\end{aligned}$$

Conclusion: If the system (14) is controllable, then we can arbitrarily assign the eigenvalues of $A + BK$ with an appropriate choice of K.

How do we know a matrix T satisfying (15) exists? Since we assumed (14) is controllable, the matrix

$$C = \begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix} \tag{17}$$

is full rank and, thus, has inverse C^{-1}. Denoting the top row of C^{-1} by $\vec{q}^T$, we note from the identity $C^{-1}C = I$ that

$$\vec{q}^TC = \begin{bmatrix} \vec{q}^TA^{n-1}B & \cdots & \vec{q}^TAB & \vec{q}^TB \end{bmatrix} = [1\ 0\ \cdots\ 0]. \tag{18}$$

We will use this equation to show that the choice

$$T = \begin{bmatrix} \vec{q}^T \\ \vec{q}^T A \\ \vdots \\ \vec{q}^T A^{n-1} \end{bmatrix}$$

indeed satisfies (15) where A_c and B_c are as in (13). The second equality in (15) follows because

$$TB = \begin{bmatrix} \vec{q}^T \\ \vec{q}^T A \\ \vdots \\ \vec{q}^T A^{n-1} \end{bmatrix} B = \begin{bmatrix} \vec{q}^T B \\ \vec{q}^T AB \\ \vdots \\ \vec{q}^T A^{n-1} B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = B_c$$

by (18). To verify the first equality in (15) note that

$$TA = \begin{bmatrix} \vec{q}^T A \\ \vec{q}^T A^2 \\ \vdots \\ \vec{q}^T A^n \end{bmatrix} \tag{19}$$

and compare this to

$$\begin{aligned} A_c T &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix} \begin{bmatrix} \vec{q}^T \\ \vec{q}^T A \\ \vdots \\ \vec{q}^T A^{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \vec{q}^T A \\ \vdots \\ \vec{q}^T A^{n-1} \\ \vec{q}^T (a_1 I + a_2 A + \cdots + a_n A^{n-1}) \end{bmatrix}. \end{aligned} \tag{20}$$

Indeed the rows of (20) and (19) match[5] and thus $TAT^{-1} = A_c$, which is the first equality in (15).

[5] The bottom rows match as a consequence of the Cayley-Hamilton Theorem that you saw in Discussion 8B. It says that a matrix satisfies its own characteristic polynomial:

$$A^n - a_n A^{n-1} - \cdots - a_2 A - a_1 I = 0.$$