EE16B - Spring'20 - Lecture 13A Notes[1]

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State Feedback Control

Suppose we are given a single-input control system

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t), \quad \vec{x}(t) \in \mathbb{R}^n, \ u(t) \in \mathbb{R}, \tag{1}$$

and we wish to bring $\vec{x}(t)$ to the equilibrium $\vec{x} = 0$ from any initial condition $\vec{x}(0)$. To do this we will use the "control policy"

$$u(t) = k_1 x_1(t) + k_2 x_2(t) + \cdots + k_n x_n(t) \tag{2}$$

where $k_1, k_2, \ldots, k_n$ are to be determined. Rewriting (2) as

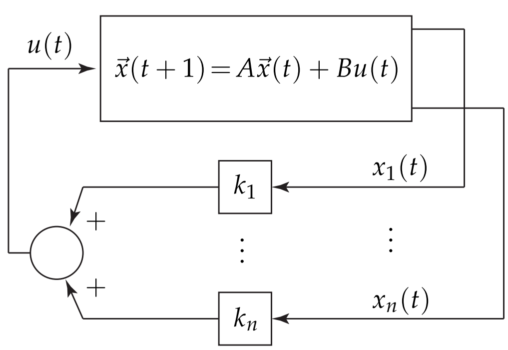
$$\boxed{u(t) = K\vec{x}(t)} \tag{3}$$

with row vector $K = [k_1 \, k_2 \, \cdots \, k_n]$, and substituting in (1), we get

$$\boxed{\vec{x}(t+1) = (A + BK)\vec{x}(t).} \tag{4}$$

Thus, if we can choose K such that all eigenvalues of $A + BK$ are inside the unit circle, $|\lambda_i(A + BK)| < 1$, $i = 1, \ldots, n$, then $\vec{x}(t) \to 0$ for any $\vec{x}(0)$ from our stability discussions in the previous lectures.

We will see that if the system (1) is controllable, then we can arbitrarily assign the eigenvalues of $A + BK$ by appropriately chosing K. Thus, in addition to bringing the eigenvalues inside the unit circle for stability, we can place them in favorable locations to shape the transients, *e.g.*, to achieve a well damped convergence.



We refer to (4) as the "closed-loop" system since the control policy (2) generates a feedback loop as depicted in the block diagram. The state variables are measured at every time step t and the input $u(t)$ is synthesized as a linear combination of these measurements.

Comparison to Open Loop Control

Recall that controllability allowed us to calculate an input sequence $u(0), u(1), u(2), \ldots$ that drives the state from $\vec{x}(0)$ to any $\vec{x}_{\text{target}}$. Thus, an alternative to the feedback control (2) is to select $\vec{x}_{\text{target}} = 0$, calculate an input sequence based on $\vec{x}(0)$, and to apply this sequence in an "open-loop" fashion without using further state measurements as depicted below.

$$\xrightarrow{u(0), u(1), u(2), \ldots} \boxed{\vec{x}(t+1) = A\vec{x}(t) + Bu(t)}$$

The trouble with this open-loop approach is that it is sensitive to uncertainties in A and B, and does not make provisions against disturbances that may act on the system.

By contrast, feedback offers a degree of robustness: if our design of K brings the eigenvalues of $A + BK$ to well within the unit circle, then small perturbations in A and B would not move these eigenvalues outside the circle. Thus, despite the uncertainty, solutions converge to $\vec{x} = 0$ in the absence of disturbances and remain bounded in the presence of bounded disturbances.

Eigenvalue Assignment by State Feedback: Examples

<u>Example 1</u>: Consider the second order system

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

and note that the eigenvalues of A are the roots of the polynomial

$$\det(\lambda I - A) = \lambda^2 - a_2\lambda - a_1.$$

If we substitute the control

$$u(t) = K\vec{x}(t) = k_1 x_1(t) + k_2 x_2(t)$$

the closed-loop system becomes

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ a_1 + k_1 & a_2 + k_2 \end{bmatrix}}_{A+BK} \vec{x}(t)$$

and, since $A + BK$ has the same structure as A with a_1, a_2 replaced by $a_1 + k_1$, $a_2 + k_2$, the eigenvalues of $A + BK$ are the roots of

$$\lambda^2 - (a_2 + k_2)\lambda - (a_1 + k_1).$$

Now if we want to assign the eigenvalues of $A + BK$ to desired values λ_1 and λ_2, we must match the polynomial above to

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2,$$

that is,

$$a_2 + k_2 = \lambda_1 + \lambda_2 \quad \text{and} \quad a_1 + k_1 = -\lambda_1\lambda_2.$$

This is indeed accomplished with the choice $k_1 = -a_1 - \lambda_1\lambda_2$ and $k_2 = -a_2 + \lambda_1 + \lambda_2$, which means that we can assign the closed-loop eigenvalues as we wish.

Example 2: Let's apply the eigenvalue assignment procedure above to

$$\vec{x}(t+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}_{A} \vec{x}(t) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(t).$$

Now we have

$$A + BK = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1 + k_1 & 1 + k_2 \\ 0 & 2 \end{bmatrix}$$

and, because this matrix is upper triangular, its eigenvalues are the diagonal entries:

$$\lambda_1 = 1 + k_1 \quad \text{and} \quad \lambda_2 = 2.$$

Note that we can move λ_1 with the choice of k_1, but we have no control over λ_2. In fact, since $|\lambda_2| > 1$, the closed-loop system remains unstable no matter what control we apply.

This is a consequence of the uncontrollability[2] of this system: the second state equation

$$x_2(t+1) = 2x_2(t)$$

can't be influenced by $u(t)$, and $x_2(t) = 2^t x_2(0)$ grows exponentially.

[2] Note that B and AB are *not* linearly independent; therefore, the system is uncontrollable.

Continuous-Time State Feedback

The idea of state feedback is identical for a continuous-time system,

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t), \quad u(t) \in \mathbb{R}.$$

To bring $\vec{x}(t)$ to the equilibrium $\vec{x} = 0$ we apply

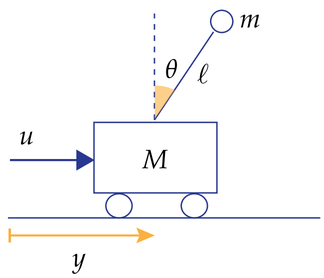
$$u(t) = K\vec{x}(t)$$

and obtain the closed-loop system

$$\frac{d}{dt}\vec{x}(t) = (A + BK)\vec{x}(t).$$

The only difference from discrete-time is the stability criterion: we must choose K such that $\mathrm{Re}(\lambda_i(A + BK)) < 0$ for each eigenvalue λ_i.

Example 3: Consider the inverted pendulum depicted below



and let the state variables be θ: the angle, $\dot{\theta}$: angular velocity, $\dot{y}$: the velocity of the cart. Then the equations of motion are

$$\begin{aligned}
\frac{d\theta}{dt} &= \dot{\theta} \\
\frac{d\dot{\theta}}{dt} &= \frac{1}{\ell(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2\ell\cos\theta\sin\theta + \frac{M+m}{m}g\sin\theta\right) \\
\frac{d\dot{y}}{dt} &= \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2\ell\sin\theta - g\sin\theta\cos\theta\right)
\end{aligned}$$

and linearization about the upright position $\theta = 0$, $\dot{\theta} = 0$, $\dot{y} = 0$ gives

$$\frac{d}{dt}\begin{bmatrix}\theta(t)\\ \dot{\theta}(t)\\ \dot{y}(t)\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1 & 0\\ \frac{M+m}{M\ell}g & 0 & 0\\ -\frac{m}{M}g & 0 & 0\end{bmatrix}}_{A}\begin{bmatrix}\theta(t)\\ \dot{\theta}(t)\\ \dot{y}(t)\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ -\frac{1}{M\ell}\\ \frac{1}{M}\end{bmatrix}}_{B}u(t).$$

We have omitted the cart position y from the state variables because we are interested in stabilizing the point $\theta = 0$, $\dot{\theta} = 0$, $\dot{y} = 0$, and we are not concerned about the final value of the position $y(t)$.

We now design a state feedback controller,

$$u(t) = k_1\theta(t) + k_2\dot{\theta}(t) + k_3\dot{y}(t).$$

Substituting the values $M = 1$, $m = 0.1$, $l = 1$, and $g = 10$, we get

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 11 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}_{B} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 11 - k_1 & -k_2 & -k_3 \\ -1 + k_1 & k_2 & k_3 \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$\lambda^3 + (k_2 - k_3)\lambda^2 + (k_1 - 11)\lambda + 10k_3 = 0$$

and, as in previous examples, we can choose k_1, k_2, k_3, to match the coefficients of this polynomial to those of

$$(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

where λ_1, λ_2, λ_3 are desired closed-loop eigenvalues.