EE16B - Spring'20 - Lecture 12A Notes[1]

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Case Study: Minimum Energy Control

In this example we review discretization, controllability, and minimum norm solutions. Consider the model of a car moving in a lane

$$\begin{aligned}\frac{dp(t)}{dt} &= v(t)\\ \frac{dv(t)}{dt} &= \frac{1}{RM}u(t)\end{aligned}$$

where $p(t)$ is position, $v(t)$ is velocity, $u(t)$ is wheel torque, R is wheel radius, and M is mass. This model is similar to an example discussed in Lecture 7A, but here we ignore friction for simplicity.

First we discretize this continuous-time model. If we apply the constant input $u(t) = u_d(k)$ from $t = kT$ to $(k+1)T$, then by integration

$$\begin{aligned}v(t) &= v(kT) + (t-kT)\frac{1}{RM}u_d(k)\\ p(t) &= p(kT) + (t-kT)v(kT) + \frac{1}{2}(t-kT)^2\frac{1}{RM}u_d(k)\end{aligned}$$

for $t \in [kT, (k+1)T)$. In particular, at $t = (k+1)T$:

$$\begin{aligned}p((k+1)T) &= p(kT) + Tv(kT) + \frac{T^2}{2RM}u_d(k)\\ v((k+1)T) &= v(kT) + \frac{T}{RM}u_d(k).\end{aligned}$$

Putting these equations in matrix/vector form and substituting $p_d(k) = p(kT)$, $v_d(k) = v(kT)$, we get

$$\begin{bmatrix}p_d(k+1)\\ v_d(k+1)\end{bmatrix} = \underbrace{\begin{bmatrix}1 & T\\ 0 & 1\end{bmatrix}}_{A}\begin{bmatrix}p_d(k)\\ v_d(k)\end{bmatrix} + \underbrace{\frac{1}{RM}\begin{bmatrix}\frac{1}{2}T^2\\ T\end{bmatrix}}_{\vec{b}}u_d(k). \qquad (1)$$

Now suppose the vehicle is at rest with $p(0) = v(0) = 0$ at $t = 0$ and the goal is to reach a target position p_{target} and stop there ($v_{\text{target}} = 0$). Recall from the lectures on controllability that if we can find a sequence $u_d(0), u_d(1), \cdots, u_d(\ell-1)$ such that

$$\begin{bmatrix}p_{\text{target}}\\ 0\end{bmatrix} = \underbrace{\begin{bmatrix}\vec{b} & A\vec{b} & \cdots A^{\ell-1}\vec{b}\end{bmatrix}}_{\mathcal{C}_\ell}\begin{bmatrix}u_d(\ell-1)\\ u_d(\ell-2)\\ \vdots\\ u_d(0)\end{bmatrix} \qquad (2)$$

then we reach the desired state in ℓ time steps, that is at time $t = \ell T$.

Since we have $n = 2$ state variables the controllability test we learned checks whether C_ℓ with $\ell = 2$ spans $\mathbb{R}^2$. This is indeed the case, since

$$C_2 = \begin{bmatrix} \vec{b} & A\vec{b} \end{bmatrix} = \frac{1}{RM} \begin{bmatrix} \frac{1}{2}T^2 & \frac{3}{2}T^2 \\ T & T \end{bmatrix}$$

has linearly independent columns.

Although this test also suggests we can reach the target state in two steps, the resulting values of $u_d(0)$ and $u_d(1)$ will likely exceed physical limits. For example, if we take the values[2] $RM = 5000$ kg m, $T = 0.1$ s, $p_{\text{target}} = 1000$ m, then

[2] say, for a sedan with mass $M \approx 1700$ kg and wheel radius $R \approx 0.3m$

$$\begin{bmatrix} u_d(1) \\ u_d(0) \end{bmatrix} = C_2^{-1} \begin{bmatrix} p_{\text{target}} \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \cdot 10^8 \\ 5 \cdot 10^8 \end{bmatrix} \text{kg}\,\text{m}^2/\text{s}^2,$$

which exceeds the torque and braking limits of a typical car by 5 orders of magnitude.[3]

[3] If our car could deliver the torque $u_d(0) = 5 \cdot 10^8$ kg m^2/s^2, then from (1) we would reach $v_d(1) = v(T) = 10^4$ m/s (22,369 mph) in $T = 0.1$ seconds!

Therefore, in practice we need to select a sufficiently large number of time steps ℓ. This leads to a wide controllability matrix C_ℓ and allows for infinitely many input sequences that satisfy (2). Among them we can select the minimum norm solution so we spend the least control energy. Using the minimum-norm solution formula

$$\begin{bmatrix} u_d(\ell-1) \\ u_d(\ell-2) \\ \vdots \\ u_d(0) \end{bmatrix} = C_\ell^T (C_\ell C_\ell^T)^{-1} \begin{bmatrix} p_{\text{target}} \\ 0 \end{bmatrix}$$

and quite a bit of algebra, one will obtain the input sequence

$$u_d(k) = \frac{6RM(\ell - 1 - 2k)}{T^2 \ell(\ell^2 - 1)} p_{\text{target}}, \quad k = 0, \cdots, \ell - 1.$$

In the plot below we show this input sequence, as well as the resulting velocity and position profiles for $RM = 5000$ kg m, $p_{\text{target}} = 1000$ m, $T = 0.1$ s, and $\ell = 1200$. With these parameters we allow $\ell T = 120$ s (2 minutes) to travel 1 km. Note that the vehicle accelerates in the first half of this period and decelerates in the second half, reaching the maximum velocity 12.5 m/s ($\approx$ 28 mph) in the middle. The acceleration and deceleration are hardest at the very beginning and at the very end, respectively. The corresponding torque is within a physically reasonable range, $[-2000, 2000]$ Nm.

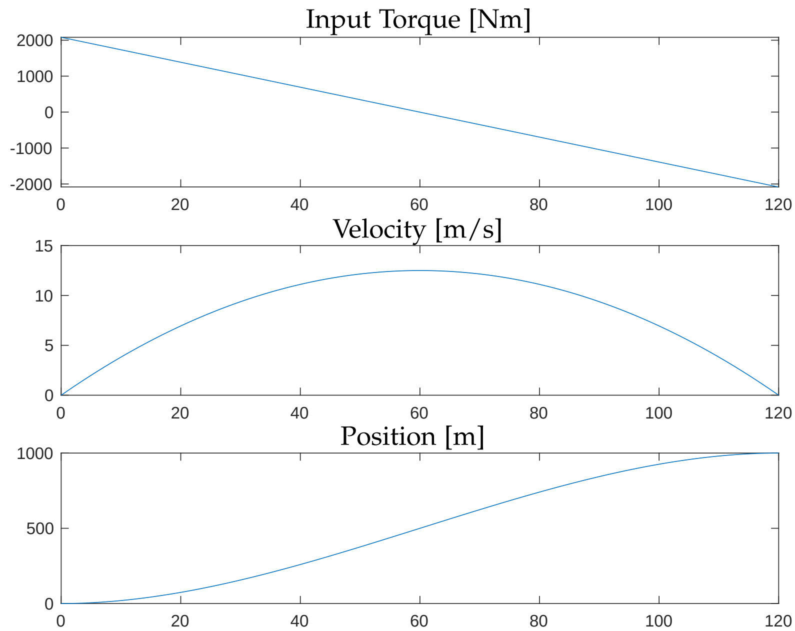

Figure 1: The minimum norm input torque sequence, and the resulting velocity and position profiles for $RM = 5000$ kg m, $p_{\text{target}} = 1000$ m, $T = 0.1$ s, and $\ell = 1200$. The horizontal axis is time, which ranges from 0 to $\ell T = 120$ s (2 minutes). The vehicle accelerates in the first half of this period and decelerates in the second half, reaching the maximum velocity 12.5 m/s ($\approx$ 28 mph) in the middle.

Stability of Linear State Models

The Scalar Case

We first study a system with a single state variable $x(t)$ that obeys

$$x(t+1) = \lambda x(t) + bu(t) \tag{3}$$

where λ and b are constants. If we start with the initial condition $x(0)$, then we get by recursion

$$\begin{aligned}
x(1) &= \lambda x(0) + bu(0) \\
x(2) &= \lambda x(1) + bu(1) = \lambda^2 x(0) + \lambda bu(0) + bu(1) \\
x(3) &= \lambda x(2) + bu(2) = \lambda^3 x(0) + \lambda^2 bu(0) + \lambda bu(1) + bu(2) \\
&\vdots \\
x(t) &= \lambda^t x(0) + \lambda^{t-1} bu(0) + \lambda^{t-2} bu(1) + \cdots + \lambda bu(t-2) + bu(t-1),
\end{aligned}$$

rewritten compactly as:

$$\boxed{x(t) = \lambda^t x(0) + \sum_{k=0}^{t-1} \lambda^{t-1-k} bu(k) \quad t = 1, 2, 3, \ldots} \tag{4}$$

The first term $\lambda^t x(0)$ represents the effect of the initial condition and the second term $\sum_{k=0}^{t-1} \lambda^{t-1-k} bu(k)$ represents the effect of the input sequence $u(0), u(1), \ldots, u(t-1)$.

Definition. We say that a system is *stable* if its state $x(t)$ remains bounded for any initial condition and any bounded input sequence. Conversely, we say it is *unstable* if we can find an initial condition and a bounded input sequence such that $|x(t)| \to \infty$ as $t \to \infty$.

It follows from (4) that, if $|\lambda| > 1$, then a nonzero initial condition $x(0) \neq 0$ is enough to drive $|x(t)|$ unbounded. This is because $|\lambda|^t$ grows unbounded and, with $u(t) = 0$ for all t, we get $|x(t)| = |\lambda^t x(0)| = |\lambda|^t |x(0)| \to \infty$. Thus, (3) is unstable for $|\lambda| > 1$.

Next, we show that $|\lambda| < 1$ guarantees stability. In this case $\lambda^t x(0)$ decays to zero, so we need only to show that the second term in (4) remains bounded for any bounded input sequence. A bounded input means we can find a constant M such that $|u(t)| \leq M$ for all t. Thus,

$$\left| \sum_{k=0}^{t-1} \lambda^{t-1-k} b u(k) \right| \leq \sum_{k=0}^{t-1} |\lambda|^{t-1-k} |b| |u(k)| \leq |b| M \sum_{k=0}^{t-1} |\lambda|^{t-1-k}.$$

Defining the new index $s = t - 1 - k$ we rewrite the last expression as

$$|b| M \sum_{s=0}^{t-1} |\lambda|^s,$$

and note that $\sum_{s=0}^{t-1} |\lambda|^s$ is a geometric series that converges to $\frac{1}{1-|\lambda|}$ since $|\lambda| < 1$. Therefore, each term in (4) is bounded and we conclude stability for $|\lambda| < 1$.

Summary: The scalar system (3) is stable when $|\lambda| < 1$, and unstable when $|\lambda| > 1$.

When λ is a complex number, a perusal of the stability and instability arguments above show that the same conclusions hold if we interpret $|a|$ as the modulus of a, that is:

$$|\lambda| = \sqrt{\text{Re}\{\lambda\}^2 + \text{Im}\{\lambda\}^2}.$$

What happens when $|\lambda| = 1$? If we disallow inputs ($b = 0$), this case is referred to as "marginal stability" because $|\lambda^t x(0)| = |x(0)|$, which neither grows nor decays. If we allow inputs ($b \neq 0$), however, we can find a bounded input to drive the second term in (4) unbounded. For example, when $\lambda = 1$, the constant input $u(t) = 1$ yields:

$$\sum_{k=0}^{t-1} \lambda^{t-1-k} b u(k) = \sum_{k=0}^{t-1} b = bt$$

which grows unbounded as $t \to \infty$. Therefore, $|\lambda| = 1$ is a precarious case that must be avoided in designing systems.

The Vector Case

When $\vec{x}(t)$ is an n-dimensional vector governed by

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t), \tag{5}$$

recursive calculations lead to the solution

$$\boxed{\vec{x}(t) = A^t\vec{x}(0) + \sum_{k=0}^{t-1} A^{t-1-k}Bu(k) \quad t = 1, 2, 3, \ldots} \tag{6}$$

where the matrix power is defined as $A^t = \underbrace{A \cdots A}_{t \text{ times}}$.

Since A is no longer a scalar, stability properties are not apparent from (6). However, when A is diagonalizable we can employ the change of variables $\vec{z} := T\vec{x}$ and select the matrix T such that

$$A_{\text{new}} = TAT^{-1}$$

is diagonal. A and A_{new} have the same eigenvalues and, since A_{new} is diagonal, the eigenvalues appear as its diagonal entries:

$$A_{\text{new}} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

The state model for the new variables is

$$\vec{z}(t+1) = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \vec{z}(t) + B_{\text{new}}u(t), \quad B_{\text{new}} = TB, \tag{7}$$

which nicely decouples into scalar equations:

$$z_i(t+1) = \lambda_i z_i(t) + b_i u(t), \quad i = 1, \ldots, n \tag{8}$$

where we denote by b_i the i-th entry of B_{new}. Then, the results for the scalar case above imply stability when $|\lambda_i| < 1$ and instability when $|\lambda_i| > 1$.

For the whole system to be stable each subsystem must be stable, therefore we need $|\lambda_i| < 1$ for each $i = 1, \ldots, n$. If there exists at least one eigenvalue λ_i with $|\lambda_i| > 1$ then we conclude instability because we can drive the corresponding state $z_i(t)$ unbounded.

Summary: The discrete-time system (5) is stable if $|\lambda_i| < 1$ for each eigenvalue $\lambda_1, \ldots, \lambda_n$ of A, and unstable if $|\lambda_i| > 1$ for some eigenvalue λ_i.