7-2, Lecture 8: Vector Differential Equations

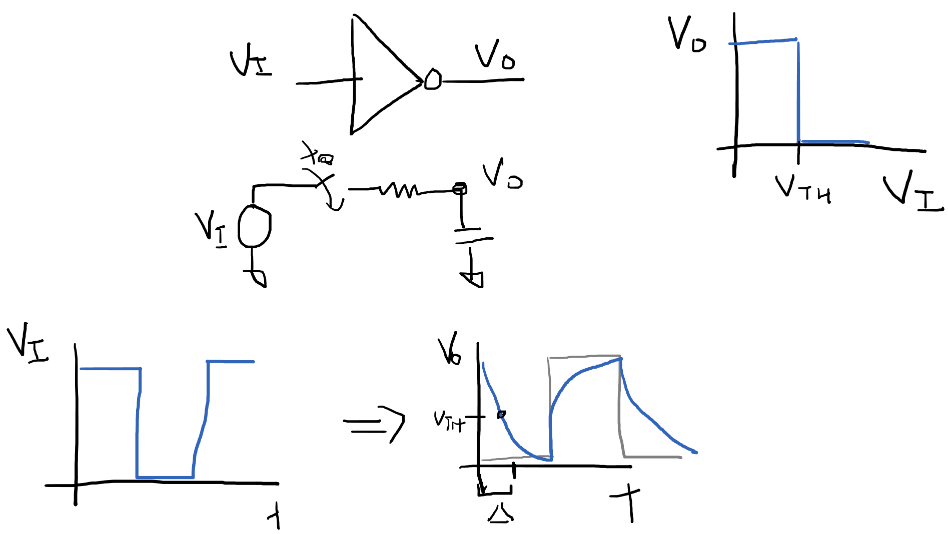
Thursday, July 2, 2020 11:08 AM

Circuits Module in Review…

I. Modeling Circuits with Differential Equations

   a. First order differential equations with constant inputs

      i. Used transistors and inverters as an application of transients (capacitors charging and discharging to 0)



      ii. Solved differential equation

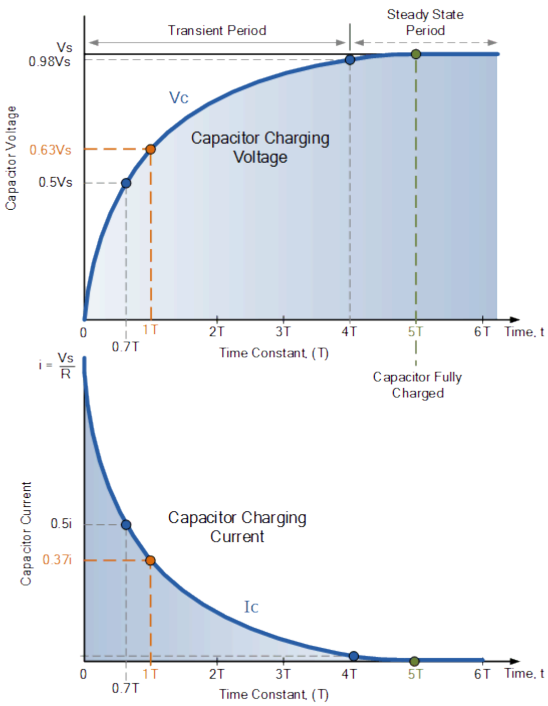
$$\frac{d}{dt}x(t) + a x(t) = b$$

$$x(t) = b/a + (x_0 - b/a)e^{-at}$$

      iii. Plugged in circuit values and looked at time constant τ to solve for delay Δ through an inverter

$$\text{Charge}: V_o(t) = VDD\left(1 - e^{-t/\tau}\right), \quad \tau = RC$$

$$\text{discharge}: V_o(t) = VDD\, e^{-t/\tau}$$



Time Constant	RC Value	Percentage of Maximum Voltage
0.5 time constant	0.5T = 0.5RC	39.3%
0.7 time constant	0.7T = 0.7RC	50.3%
1.0 time constant	1T = 1RC	63.2%
2.0 time constants	2T = 2RC	86.5%
3.0 time constants	3T = 3RC	95.0%
4.0 time constants	4T = 4RC	98.2%
5.0 time constants	5T = 5RC	99.3%

b. First order differential equations with time varying inputs

i. Solved differential equation

$$\frac{d}{dt} x(t) + a x(t) = g(t)$$

$$x(t) = e^{-at} \int_{t_0}^{t} g(\tau) e^{a\tau} d\tau + x_0 e^{-at}$$

ii. Plugged in circuit values and choose $g(t) = \tilde{V} e^{st}$ for eigenfunction properties with derivative operator

$$\frac{d}{dt} e^{st} = s e^{st}$$

$$V_o(t) = \underbrace{\frac{\tilde{V}_i}{1 + sRC} e^{st}} + \underbrace{\left(V_o(0) - \frac{\tilde{V}_i}{1 + sRC}\right) e^{-t/RC}}$$

iii. Choose s=jw to make e^st term periodic (as per Euler's Formula)

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t)$$ (proved using Taylor Expansion)

If we wait long enough in RC time constants, the $e^{-t/RC}$ term will decay, leaving a

If we wait long enough in RC time constants, the $e^{-t/RC}$ term will decay, leaving a periodic steady state of just the $e^{j\omega t}$ term

II. Phasor Domain

a. Defined interesting input signals in $e^{j\omega t}$ basis

$$r\cos(\omega t+\phi) = \frac{1}{2}\left(\tilde{V}e^{j\omega t} + \overline{\tilde{V}e^{j\omega t}}\right), \quad \tilde{V} = re^{j\phi}$$

$$r\sin(\omega t+\phi) = \frac{1}{2j}\left(\tilde{V}e^{j\omega t} - \overline{\tilde{V}e^{j\omega t}}\right)$$

Music/bio sensor/wireless signal $= r_1\cos(\omega_1 t+\phi_1) + r_2\cos(\omega t+\phi_2) + \dots$

b. Navigating Complex Plane with Phasors

Phasor is a complex number

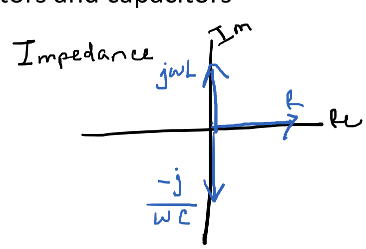
$re^{j\phi}$, magnitude r & angle ϕ

Used to scale or rotate another phasor or function or time

$re^{j\phi}\cdot qe^{j\theta}$ $\qquad\qquad$ $re^{j\phi}\cdot e^{j\omega t}$

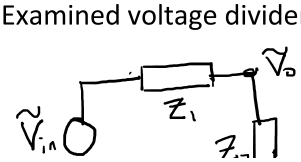
c. Used Phasor Domain to write complex impedances for inductors and capacitors

$Z_R = R \qquad Z_L = j\omega L \qquad Z_C = \frac{1}{j\omega C}$

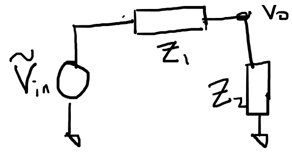


d. Used the frequency dependent impedances Z_L and Z_C to make first order frequency filters

i. Examined voltage divider transfer function



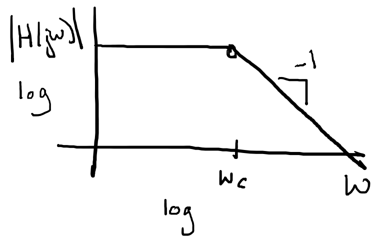
$$H(j\omega) = \frac{\tilde{V}_o}{\tilde{V}_{in}} = \frac{Z_1}{Z_1+Z_2}$$

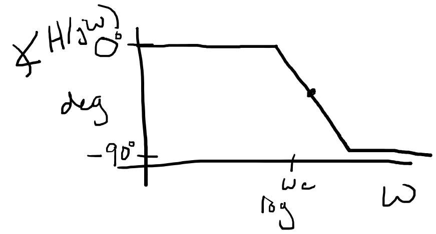


$$H(j\omega) = \frac{\tilde{V}_o}{\tilde{V}_{in}} = \frac{Z_1}{Z_1 + Z_2}$$

ii. Plotted magnitude $|H(j\omega)|$ and angle $\measuredangle H(j\omega)$ with bode plots

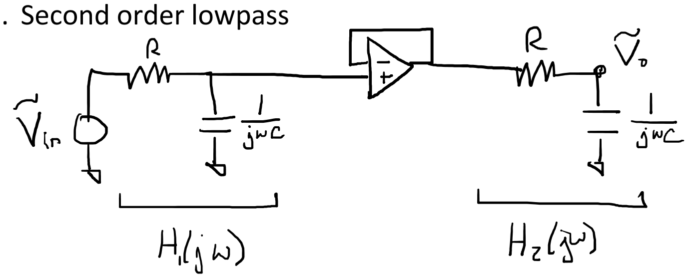
Ex. $H(j\omega) = \dfrac{1}{1 + j\omega RC}$





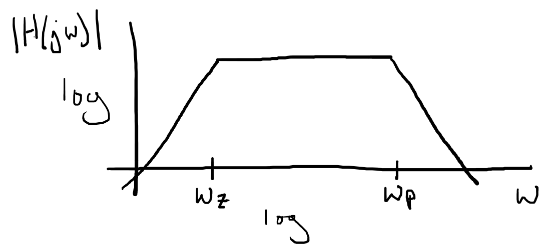
e. Made second order filters to filter interferers and noise better

i. Second order lowpass



$$H(j\omega) = H_1(j\omega)\,H_2(j\omega) = \frac{1}{(1 + j\omega RC)^2}$$

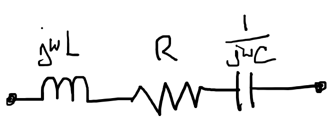
ii. Second order bandpass



$$H(j\omega) = \frac{j\omega/\omega_z}{(1 + j\omega/\omega_z)} \cdot \frac{1}{(1 + j\omega/\omega_p)}$$

F. Cancelled Imaginary Impedances at the Resonant Frequency

Z_{LRC}:



$$Z_{LRC} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

Z_{LRC} Purely real at $\omega_r L = \dfrac{1}{\omega_r C}$

$$\omega^2 = \frac{1}{LC}$$

Today

I. Representing more function in e^jwt basis

II. Back to diff eqns

a. Second order in time domain
b. Diagonal matrices
c. Diagonalization with eigen decomposition
d. Stability and eigenvalues

I. Representing more function in e^jwt basis

Use linear combinations of e^jwt to represent cosines, sines, music, wireless communication

Square wave?

```
y = sin(t);
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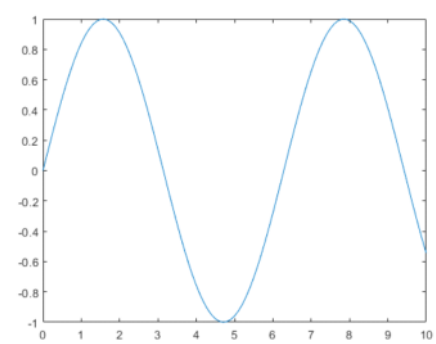
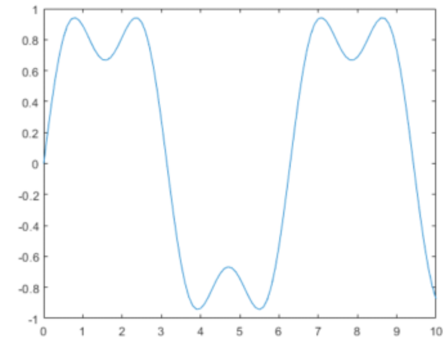
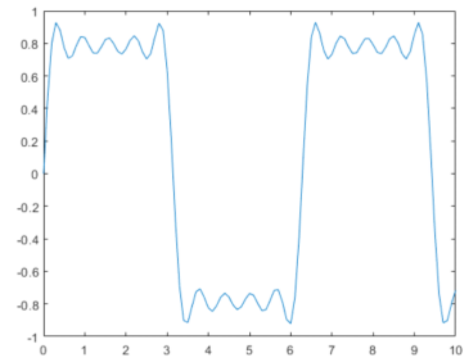


`y = sin(t) + sin(3*t)/3;`



`y = sin(t) + sin(3*t)/3 + sin(5*t)/5 + sin(7*t)/7 + sin(9*t)/9;`



Much like a Taylor Series, can represent sq wave, but I need infinite sines to do it

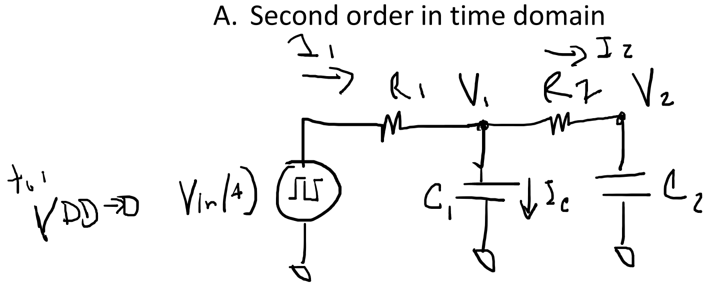
Could put a sq wave into phasor domain and look at the effect transfer function

Would need infinite sines to do this

$\Rightarrow$ Phasor domain is not always the best way to examine functions like a sq wave

II. Back to Differential Equations

A. Second order in time domain



$$I_1 = I_C + I_2$$

$$I_1 = \frac{V_{in} - V_1}{R_1}$$

$$I_2 = V_1 - V_2$$

$V_{in}(0) = 0$

$V_2(0) = V_1(0) = VDD$

$$\frac{\quad}{R_2} = \frac{d}{dt} V_2 C_2$$

$$I_c = \frac{d}{dt} V_1 C_1$$

$$\frac{d}{dt} V_1 = \frac{I_1}{C_1} - \frac{I_2}{C_1} = \frac{\cancel{V_{in}}^{0} - V_1}{R_1 C_1} - \frac{V_1 - V_2}{R_2 C_1}$$

$$\left[\begin{aligned} \frac{d}{dt} V_1 &= \frac{V_2}{R_2 C_1} - V_1\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) \\ \frac{d}{dt} V_2 &= \frac{V_1}{R_2 C_2} - \frac{V_2}{R_2 C_2} \end{aligned}\right.$$

second order diff eqn

matrix representation

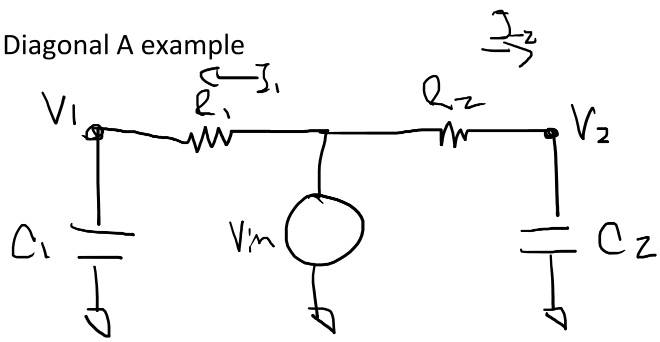
$$\left(\frac{d}{dt}\right)^2 x(t) + a\frac{d}{dt} x(t) + b\, x(t) = C$$

$$\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} & A & \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} -\left(\frac{1}{R_1 C_1} + \frac{1}{R_2 C_1}\right) & \frac{1}{R_2 C_1} \\ \frac{1}{R_2 C_2} & \frac{-1}{R_2 C_2} \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\frac{d}{dt}\vec{x} = A\vec{x} \quad \leftarrow \quad e^{ct}$$ somewhere involved

B. Diagonal A example



$V_{in}: VDD \to 0$

$V_{in}(0) = 0$

$V_2(0) = V_1(0) = VDD$

$$\frac{\cancelto{0}{V_{in}} - V_1}{R_1} = I_1 = \frac{d}{dt} V_1 C_1$$

$$\frac{\cancelto{0}{V_{in}} - V_2}{R_2} = I_2 = \frac{d}{dt} V_2 C_2$$

$$\Rightarrow \quad \frac{d}{dt} V_1 = \frac{-V_1}{R_1 C_1} \qquad \frac{d}{dt} V_2 = \frac{-V_2}{R_2 C_2}$$

$$\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{R_1 C_1} & 0 \\ 0 & \frac{-1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$V_1(t) = C_1 e^{\lambda_1 t} \qquad \lambda_1 = \frac{-1}{R_1 C_1}$$

$$V_2(t) = C_2 e^{\lambda_2 t} \qquad \lambda_2 = \frac{1}{R_2 C_2}$$

Want: apply some transform to our matrices

$$\begin{array}{ccc} \vec{x} & \xrightarrow{A} & A\vec{x} \\ B \downarrow & & \uparrow B^{-1} \end{array}$$

$$
\begin{tikzcd}
{} \arrow[d, "B"'] & {} \arrow[d, dash, "B'"] \\
\vec{x} \arrow[r, "D"'] & D\vec{x}
\end{tikzcd}
$$

convient basis
where D is diagonal

C. Diagonalization with Eigendecompostion

Eigenvalues & Eigenvectors

$$A\vec{v}_n = \lambda_n \vec{v}_n$$

A has eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$

and eigenvectors $\vec{v}_1, \vec{v}_2 \quad \ldots \quad \vec{v}_n$

$$\begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1k} \end{bmatrix}, \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2k} \end{bmatrix}, \cdots \begin{bmatrix} x_{k1} \\ x_{k2} \\ \vdots \\ x_{kk} \end{bmatrix}.$$

Create a matrix V of eigenvectors

$$V = [\vec{v}_1 \ \vec{v}_2 \ \cdots \vec{v}_n]$$
$$\cancel{= [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_k]}$$
$$= \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{k1} \\ x_{12} & x_{22} & \cdots & x_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k} & x_{2k} & \cdots & x_{kk} \end{bmatrix}$$

$$AV = A[\vec{v}_1 \ \vec{v}_2 \ \cdots \ \vec{v}_n]$$
$$= [A\vec{v}_1 \ A\vec{v}_2 \ \cdots \ A\vec{v}_n]$$

$$= \begin{bmatrix} \lambda_1 \vec{v}_1 & \lambda_2 \vec{v}_2 & \cdots & \lambda_n \vec{v}_n \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_{11} & \lambda_2 x_{21} & \cdots & \lambda_k x_{k1} \\ \lambda_1 x_{12} & \lambda_2 x_{22} & \cdots & \lambda_k x_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 x_{1k} & \lambda_2 x_{2k} & \cdots & \lambda_k x_{kk} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{k1} \\ x_{12} & x_{22} & \cdots & x_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1k} & x_{2k} & \cdots & x_{kk} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{bmatrix}$$

$$AV = V\Lambda$$

$$\begin{tikzcd} \vec{x} \arrow[r, dashed] \arrow[d, "V^{-1}"'] & A\vec{x} \\ \tilde{\vec{x}} \arrow[r, "\Lambda"'] & \Lambda\tilde{\vec{x}} \arrow[u, "V"'] \end{tikzcd}$$

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

$$\left[\frac{d}{dt}\tilde{\vec{x}} = \Lambda\tilde{\vec{x}} \right]$$

$$\tilde{\vec{x}} = V^{-1}\vec{x} \quad , \quad \vec{x} = V\tilde{\vec{x}}$$

In our convenient domain

$$\frac{d}{dt}\tilde{\vec{x}} = \Lambda\tilde{\vec{x}} \implies \tilde{\vec{x}} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

solve for constants $c_1, c_2 \cdots c_n$ with initial conds $\tilde{\vec{x}}(0) = V\vec{x}(0)$

$$\vec{x} = V\tilde{\vec{x}}$$

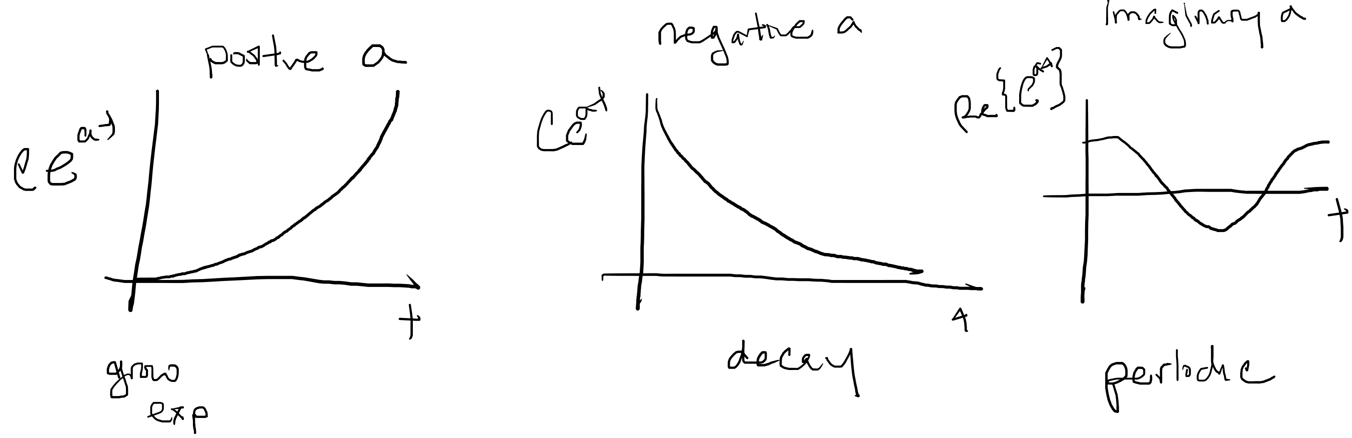
$$\vec{x} = \begin{bmatrix} c'_{11} e^{\lambda_1 t} + c'_{12} e^{\lambda_2 t} + \cdots c'_{1n} e^{\lambda_n t} \\ c'_{21} e^{\lambda_1 t} + \cdots \end{bmatrix}$$

$$\begin{bmatrix} C'_{n} e^{\lambda t} + \cdots \\ \vdots \\ C''_{in} e^{\lambda t} + \cdots \end{bmatrix}$$

could solve for constants here

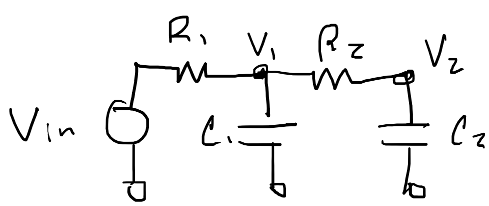
D. Stability and Eigen Values

i. look at Ce^{at} for various a



Look at eigenvalues of matrix A and know if if x is going to grow, decay, or oscillate

ii. Solve for eigenvalues of our second order RC



$$A = \begin{bmatrix} -\left(\frac{1}{RC_1} - \frac{1}{R_2C_2}\right) & \frac{1}{R_2C_2} \\ \frac{1}{R_2C_2} & \frac{-1}{R_2C_2} \end{bmatrix}$$

Choose R & C

$$A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$$

Solve for the roots of the characteristic eqn

$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$

$$(A - \lambda_1 I)\vec{v}_1 = 0$$

$$\Rightarrow \det(A - \lambda I) \quad \text{find the roots}$$

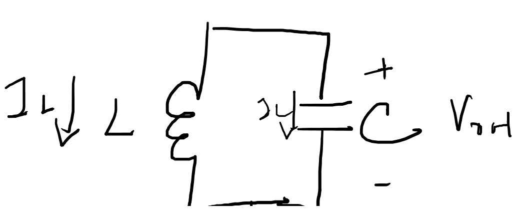
$$\det\left(\begin{bmatrix} \lambda+5 & 2 \\ 2 & \lambda+2 \end{bmatrix}\right) = (\lambda+5)(\lambda+2) - 4$$
$$= \lambda^2 + 7\lambda + 6$$
$$= (\lambda+6)(\lambda+1)$$

roots: $\lambda_{1,2} = -6, -1$

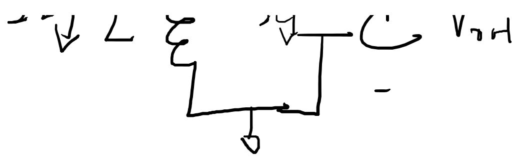
Expect

$$\vec{x} = \begin{bmatrix} c_0' e^{-6t} + c_1 e^{-t} \\ c_2 e^{-6t} + c_3 e^{-t} \end{bmatrix} \leftarrow \text{Voltages will decay}$$

iii. Eigenvalues for LC



$$I_L = -I_C$$
$$V_L = V_C = V_{out}$$



$V_L = V_C = V_{out}$

$$\begin{cases} \frac{d}{dt} I_L L = V_{out} \\ \frac{d}{dt} V_{out} C = -I_L \end{cases}$$

$$\frac{d}{dt}\begin{bmatrix} V_{out} \\ I_L \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}}_{A} \begin{bmatrix} V_{out} \\ I_L \end{bmatrix}$$

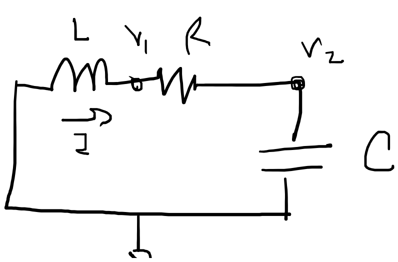
$$\det(A) = \begin{bmatrix} -\lambda & \frac{-1}{C} \\ \frac{1}{L} & -\lambda \end{bmatrix} = \lambda^2 + \frac{1}{LC}$$ ← solve for the roots

$$\lambda_{1,2} = 0 \pm j\frac{1}{\sqrt{LC}}$$ ← imaginary roots

$$\vec{x} = \begin{bmatrix} V_{out} \\ I_L \end{bmatrix}$$ ← oscillate

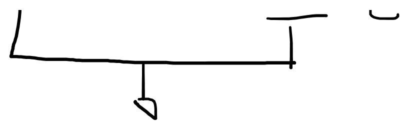
- LC tank, used to make oscillator

Iii. Eigenvalues of LCR



① $V_1 - V_2 = RI$

$$\frac{d}{dt}(V_2 - 0)\, C = I$$



$$\frac{d}{dt}(V_2 - 0)\,C = I$$

$$(0 - V_1) = \frac{d}{dt} I L$$

$$\frac{d}{dt} V_2 = \frac{1}{C} I$$

$$\frac{d}{dt} I = \frac{-RI - V_2}{L}$$

$$\frac{d}{dt}\begin{bmatrix} V_2 \\ I \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{C} \\ \frac{-1}{L} & -\frac{R}{L} \end{bmatrix}}_{A} \begin{bmatrix} V_2 \\ I \end{bmatrix}$$

$$\det\left(\begin{bmatrix} -\lambda & \frac{1}{C} \\ \frac{-1}{L} & -\frac{R}{L} - \lambda \end{bmatrix}\right) = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC}$$

$$\text{roots} \Rightarrow \lambda_{1,2} = \frac{-1}{2}\frac{R}{L} \pm \underbrace{\sqrt{\left(\frac{1}{2}\frac{R}{L}\right)^2 - \frac{1}{LC}}}_{\text{potential for imaginary values}}$$

Case 1.

$$\left(\frac{1}{2}\frac{R}{L}\right)^2 = \frac{1}{LC} \Rightarrow \lambda_{1,2} = \frac{-1}{2}\frac{R}{L} \quad \leftarrow \text{decay}$$

Case 2.

$$\left(\frac{1}{2}\frac{R}{L}\right)^2 > \frac{1}{LC} \Rightarrow \lambda_{1,2} = \text{distinct real valued} \quad \leftarrow \text{decay or grow}$$

Case 3.

$$\left(\frac{1}{2}\frac{R}{L}\right)^2 < \frac{1}{LC} \Rightarrow \lambda_{1,2} = \underbrace{-\frac{1}{2}\frac{R}{L}}_{\text{real}} \pm j\underbrace{\sqrt{\frac{1}{LC} - \left(\frac{1}{2}\frac{R}{L}\right)^2}}_{\text{imaginary}}$$

decay & oscillation

ringing