6-30, Lecture 6: Frequency Response and Bode Plots

Tuesday, June 30, 2020 10:58 AM

(Post lecture notes in purple)
(Impt equations boxed in green)

Used Taylor expansion to prove Euler's Formula

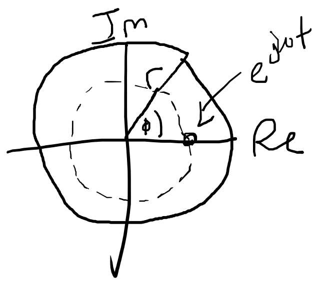
$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

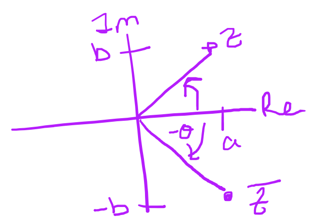
conjugate:

$$\overline{a+jb} = a - jb$$

Navigating the complex plane using Phasors

$$re^{j\phi}$$





Useful for defining magnitude and phase of our input function e^jwt

$$V_{in}(t) = \underbrace{re^{j\phi}}_{\text{phasor}} \cdot e^{j\omega t}$$

angular freq

$$\omega = 2\pi f \quad (\text{rad/sec})$$

Learned useful linear combinations of e^jwt

$$r\cos(\omega t + \phi) = \frac{1}{2}\left(re^{j\phi}e^{j\omega t} + \overline{re^{j\phi}e^{j\omega t}}\right)$$

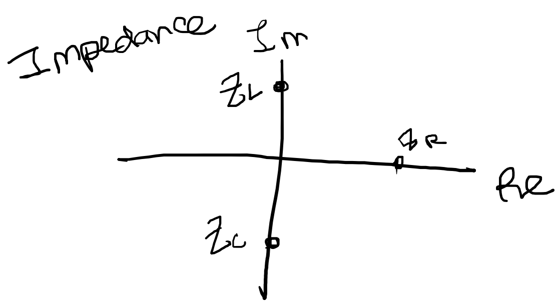
$$r\sin(\omega t + \phi) = \frac{1}{2j}\left(re^{j\phi}e^{j\omega t} - \overline{re^{j\phi}e^{j\omega t}}\right)$$

Work phasor domain

$$\tilde{V}_{in} \longrightarrow \boxed{\text{circuit}} \longrightarrow |H(j\omega)| e^{j\angle H(j\omega)} \tilde{V}_{in}$$

$$\boxed{Z_R = R \qquad Z_L = j\omega L \qquad Z_C = \frac{1}{j\omega C}}$$

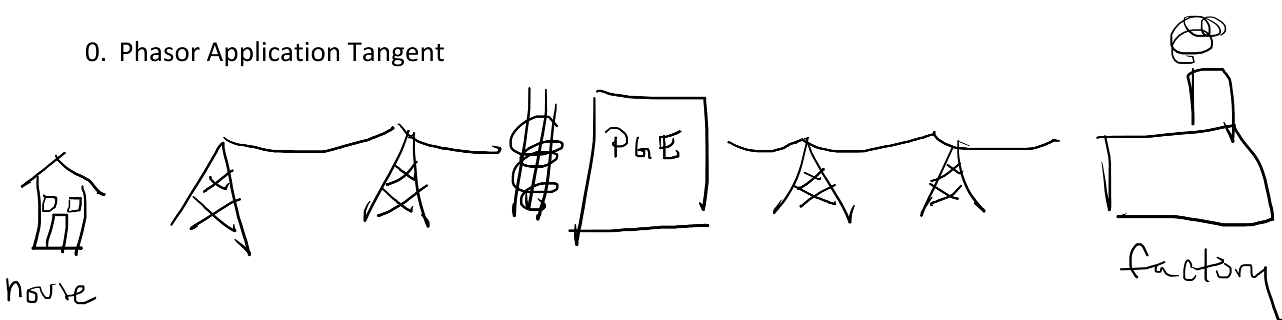
$\underbrace{\text{Reactance}}_{Im}$ and $\underbrace{\text{Resistance}}_{Re}$ of Complex Impedance



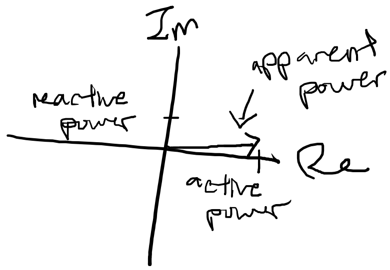
Today:

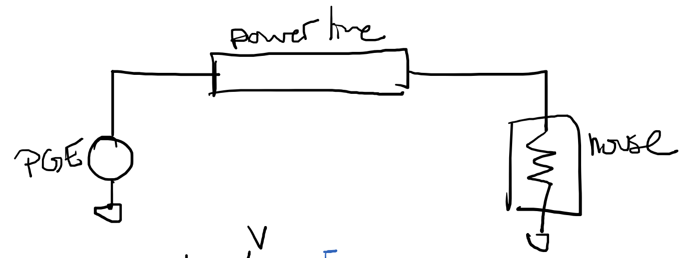
0. Phasor Application Tangent
I. Transfer Function Example
II. Frequency Response
   a. RC circuit example
   b. Passive devices over frequency
      i. Voltage divider reminder
      ii. Low Pass Filter
      iii. High Pass Filter
      iv. Active Filter
III. Bode Plots
   a. Design and plot a filter
   b. Bode plot rules
   c. ~~Application tangent: Headphone Freq Resp~~

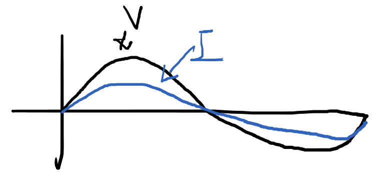
0. Phasor Application Tangent



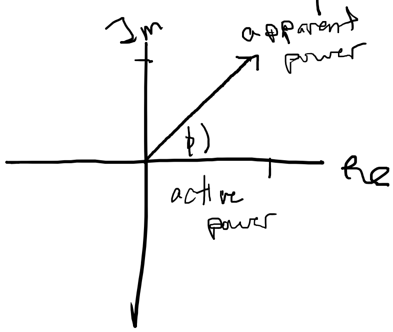
Power to house

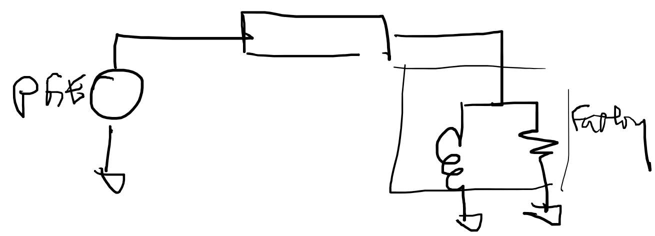


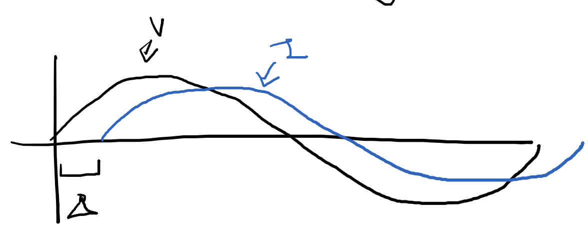




Power to Factory

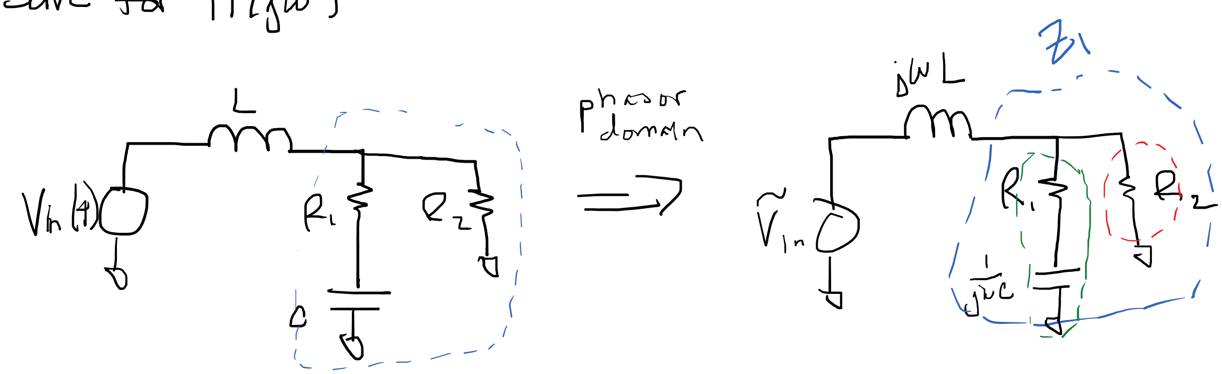






I. Transfer Function Examples

Solve for $H(j\omega)$



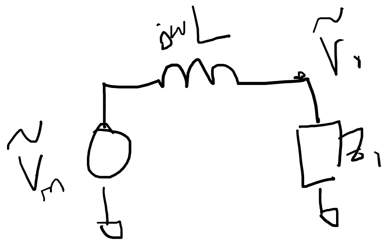
Parallel/ Series Imp Combination
Thevenin and Norton
KVL/KCL
Voltage Divider

$$Z_1 = \left(R_1 + \frac{1}{j\omega C}\right) \parallel R_2$$

Thevenin and Norton
KVL/KCL
Voltage Divider

$$Z_1 = \left(R_1 + \frac{1}{j\omega C}\right) \parallel R_2$$

$$= \frac{\left(R_1 + \frac{1}{j\omega C}\right) R_2}{\left(R_1 + \frac{1}{j\omega C}\right) + R_2}$$



$$\tilde{V}_o = \tilde{V}_{in} \frac{Z_1}{Z_1 + j\omega L}$$

$$\boxed{H(j\omega) = \frac{\tilde{V}_o}{\tilde{V}_{in}}} = \frac{Z_1}{Z_1 + j\omega L}$$

$$\boxed{H(j\omega) = |H(j\omega)| \, e^{j\measuredangle H(j\omega)}}$$

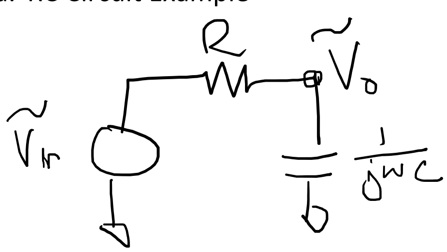
Given ω, R_1, R_2, C, L

$|H(j\omega)|$ = numpy.absolute $(H(j\omega))$

$\measuredangle H(j\omega)$ = numpy.angle $(H(j\omega))$

II. Frequency Response

a. RC Circuit Example



$$H(j\omega) = \frac{\tilde{V}_o}{\tilde{V}_{in}} = \frac{1}{1 + j\omega RC}$$

$$\tilde{V}_o = \tilde{V}_{in} H(j\omega)$$

Input 1:

$$V_{in} = \underline{1} e^{j\omega_1 t} \longrightarrow \boxed{H(j\omega)} \longrightarrow \frac{1}{1 + j\omega_1 RC} \cdot 1 \cdot e^{j\omega_1 t}$$

↑
Input
phasor

Input 2:

$$V_{in} = \tilde{V}\cos(\omega_1 t) \rightarrow \boxed{H(j\omega)} \rightarrow \frac{1}{2}\left(\frac{1}{1+j\omega_1 RC}\cdot\tilde{V}e^{j\omega_1 t} + \overline{\frac{1}{1+j\omega_1 RC}\cdot\tilde{V}e^{j\omega_1 t}}\right)$$

$$\tilde{V}\cos(\omega_1 t) = \frac{1}{2}\left(\tilde{V}e^{j\omega_1 t} + \overline{\tilde{V}e^{j\omega_1 t}}\right)$$

Input 3

$$V_{in} = \tilde{V}\left(e^{j\omega_1 t} + e^{j\omega_2 t} + \dots\right) \rightarrow \boxed{H(j\omega)} \rightarrow \tilde{V}\left(\frac{1}{1+j\omega_1 RC}e^{j\omega_1 t} + \frac{1}{1+j\omega_2 RC}e^{j\omega_2 t}\right)$$

$H(j\omega)$ is freq dependent

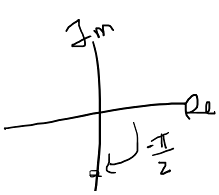
Ex

$R = 100 \quad C = 1\times 10^{-6} \quad \omega_1 = 1 \quad \omega_2 = 1\times 10^{9}$

$$H(j\omega_1) = \frac{1}{1 + j\,1(100)(1\times 10^{-6})} = \frac{1}{1 + \cancel{j\,1\times 10^{-4}}^{\,\to 0}} \cong 1$$

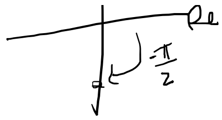
$$H(j\omega_2) = \frac{1}{\cancel{1}+ j\,1\times 10^{5}} \cong \frac{1}{j\,1\times 10^{5}} \cong \frac{1}{j}0 = -j0$$

$$V_o(t) = \tilde{V}\underbrace{H(j\omega_1)}_{\sim 1}e^{j\omega_1 t} + \tilde{V}\underbrace{H(j\omega_2)}_{\sim -j0}e^{j\omega_2 t}$$



$|H(j\omega)| = 1 \qquad |H(j\omega)| = 0$

$\measuredangle H(j\omega) = 0 \qquad \measuredangle H(j\omega) = -\frac{\pi}{2}$



$\measuredangle H(j\omega) = 0$ $\measuredangle H(j\omega) = -\frac{\pi}{2}$

small ω Large ω

Low Pass Filter

b. Passive devices over frequency

Resistor

$Z_R = R$

$$Z_R @ (\omega = 0) = R$$

$$Z_R @ (\omega = \infty) = R$$

Capacitor

$Z_C = \frac{1}{j\omega C}$

$$Z_C @ (\omega = 0) = \frac{1}{j \cdot 0 \cdot C} = -j\infty \text{ (open)}$$

$$Z_C @ (\omega = \infty) = \frac{1}{j \infty C} = -j0 \text{ (short)}$$

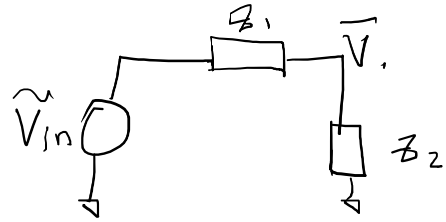
Inductor

$Z_L = j\omega L$

$$Z_L @ (\omega = 0) = j \cdot 0 \cdot L = j0 \text{ (short)}$$

$$Z_L @ (\omega = \infty) = j \cdot \infty \cdot L = j\infty \text{ (open)}$$

i. Voltage divider reminder

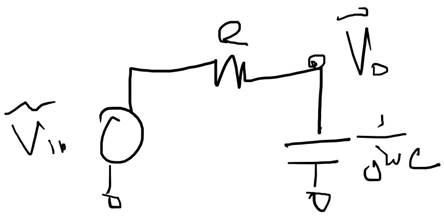


$$\boxed{\frac{\tilde V_{Z1}}{\tilde V_{Z2}} = \frac{Z_1}{Z_2}}$$

$\tilde V_o$ large for $Z_2 \gg Z_1$

$\tilde V_o$ small for $Z_2 \ll Z_1$

ii. Low Pass Filter

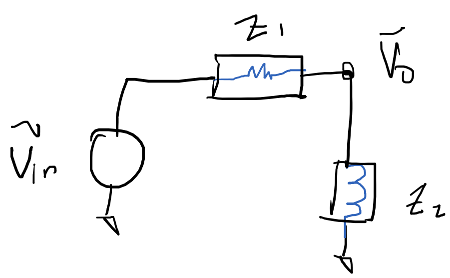


$\tilde V_o$ large $\frac{1}{j\omega C} \gg R$ ← small freq

$\tilde V_o$ small $\frac{1}{j\omega C} \ll R$ ← large freq

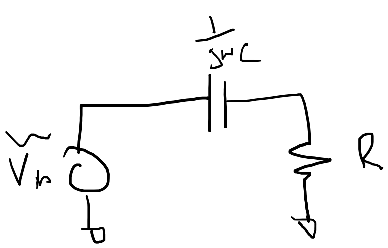
Low Pass Filter

iii. High Pass Filter



$Z_1 \gg Z_2$ small freq

$Z_1 \ll Z_2$ large freq

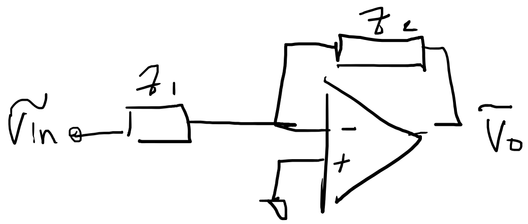


$$\tilde V_o = \tilde V_{in} \frac{R}{R + \frac{1}{j\omega C}} \Rightarrow H(j\omega) = \frac{j\omega CR}{1 + j\omega CR}$$

small ω $\rightarrow 0j$

large ω $\rightarrow 1$

iv. Active Filter



$$A_v = \frac{\tilde{V}_o}{\tilde{V}_{in}} = -\frac{Z_2}{Z_1}$$

$Z_2 = j\omega L$

$Z_1 = R$

$Z_2 >> Z_1$ large freq

$Z_2 << Z_1$ small freq

Active High Pass Filter

III. Bode Plots

a. Design and plot a filter

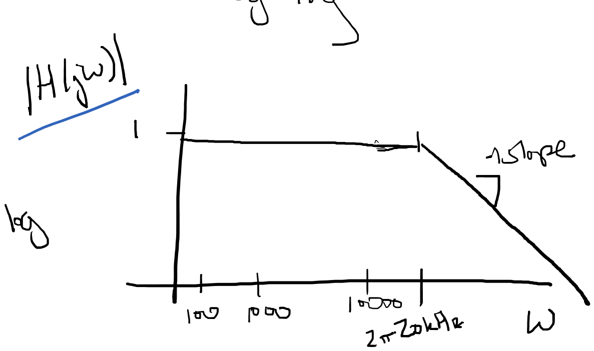
Filter out freq. above human hearing

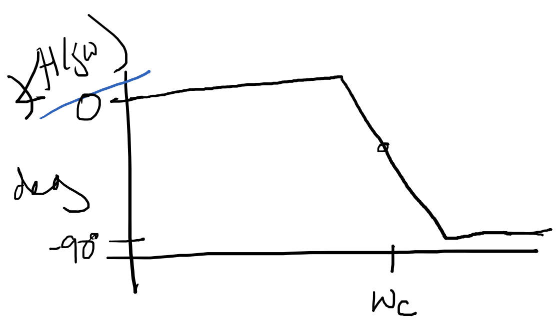
20Hz - 20kHz : $H(j\omega) \simeq 1$

Low Pass Filter

$>$ 20kHz : $H(j\omega) \to -j0$

log-log





log ↑ cutoff freq (ω_c)

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

$$|j\omega RC| = 1$$

$\omega_c = 2\pi \cdot 20kHz$

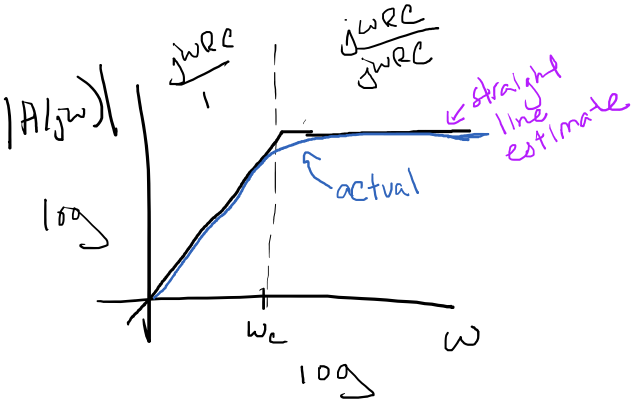
$\omega_c = \frac{1}{RC}$ ← cutoff freq, corner, pole

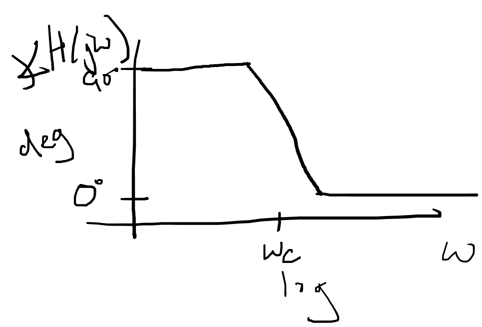
High Pass Filter

$$H(j\omega) = \frac{j\omega RC}{1 + j\omega RC}$$

← grows w/ ω

← 1 dominates until $\omega = 1/RC$



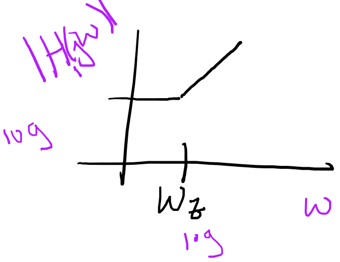


b. Bode Plot Rules

$$H(j\omega)=\frac{(j\omega)^n\left(1+j\omega/\omega_{z1}\right)\left(1+j\omega/\omega_{z2}\right)\cdots}{(j\omega)^m\left(1+j\omega/\omega_{p1}\right)\left(1+j\omega/\omega_{p2}\right)\cdots}\quad\begin{matrix}\leftarrow \text{zeros} \quad \omega=\omega_{z1},\omega_{z2},\dots \\[2ex] \leftarrow \text{poles} \quad \omega=\omega_{p1},\omega_{p2},\dots\end{matrix}$$

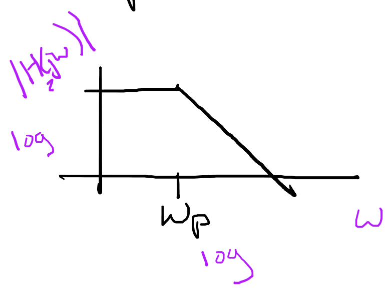
Factored transfer function

zeros



×

poles



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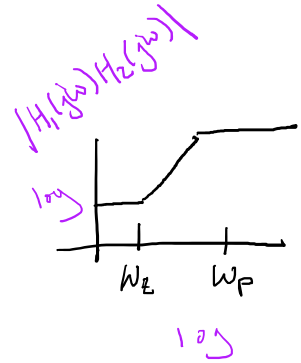
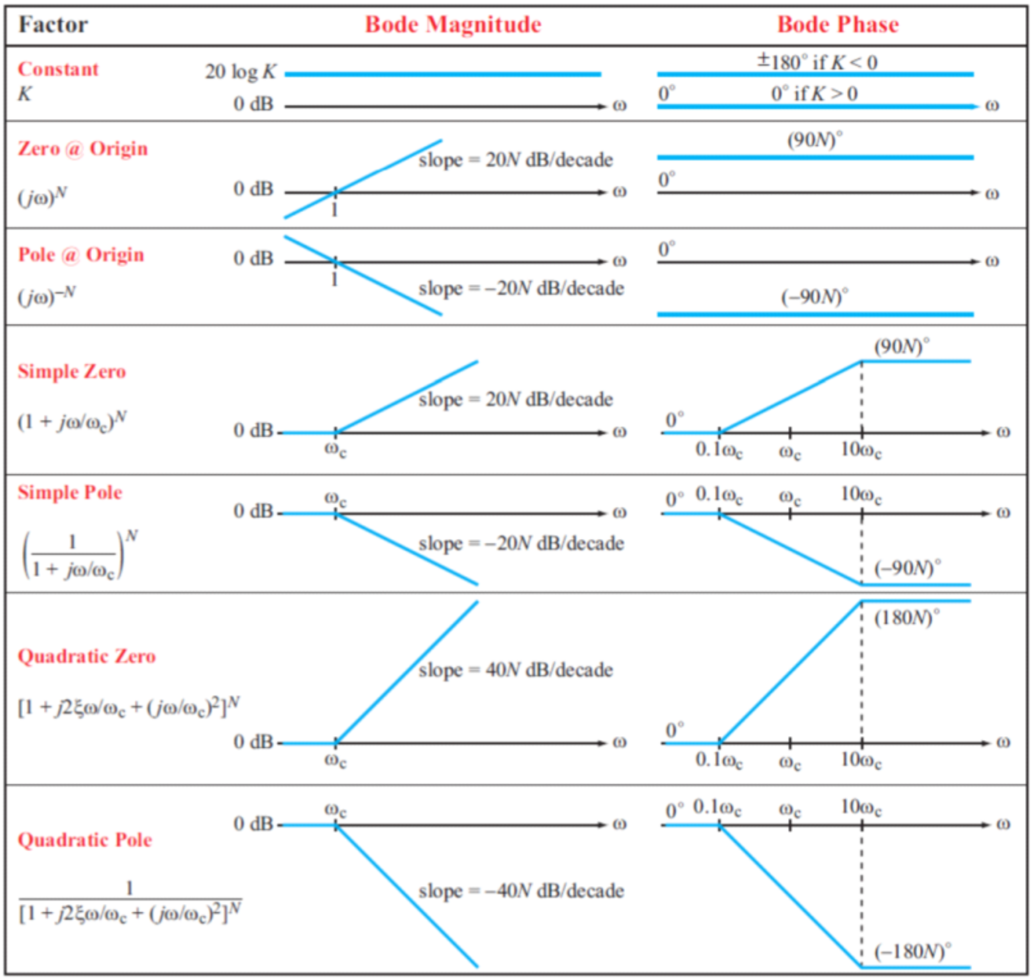


GO TO DICUSSION FOR MORE BODE PLOT PRACTICE


Factor
Bode Magnitude
Bode Phase
Constant
K
20 log K
0 dB
±180° if K < 0
0° if K > 0
Zero @ Origin
(jω)^N
slope = 20N dB/decade
(90N)°
Pole @ Origin
(jω)^−N
slope = −20N dB/decade
(−90N)°
Simple Zero
(1 + jω/ω_c)^N
(90N)°
0.1ω_c
ω_c
10ω_c
Simple Pole
(1/(1 + jω/ω_c))^N
(−90N)°
Quadratic Zero
[1 + j2ξω/ω_c + (jω/ω_c)²]^N
slope = 40N dB/decade
(180N)°
Quadratic Pole
1/[1 + j2ξω/ω_c + (jω/ω_c)²]^N
slope = −40N dB/decade
(−180N)°