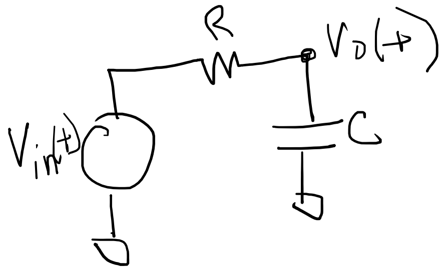
6-29, Lecture 5: Complex Plane and Phasors

Monday, June 29, 2020 10:45 AM

Last time on EE16B....

(Post lecture notes in purple)
(Impt equations boxed in green)

Looked at time varying inputs



$$\frac{d}{dt}x(t) + a x(t) = g(t), \quad x(t_0) = x_0$$

$$x(t) = e^{-at}\int_{t_0}^{t} g(\tau) e^{a\tau}\, d\tau + x_0 e^{-at}$$

$$\Rightarrow V_o(t) = e^{-t/RC}\int_{t_0}^{t} \frac{V_{in}(\theta)}{RC} e^{\theta/RC}\, d\theta + V_o(0) e^{-t/RC}$$

Plugged in functions for Vout(t)

Constant : $V_{in}(t) = V_{DD}$ ← expected transient

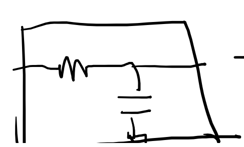
Sinusoid $= \sin(t)$ ← too hard

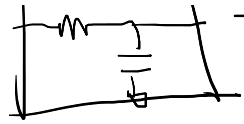
exponential $= e^{st}$ ← eigen function (steadystate)

Steady State: for convient inital conditions

$$\underbrace{V_o(t) = \frac{\tilde{V}_i}{sRC+1} e^{st}}_{\text{eigen function}} + \overbrace{\left(V_o(0) - \frac{\tilde{V}_i}{sRC+1}\right) e^{-t/RC}}^{=0}$$

Work in e^st world

$V_{in}(t) \rightarrow$  $\rightarrow \lambda V_{in}(t)$

$V_{in}(t) \rightarrow$  $\rightarrow \sim V_{in}(t)$

In e^st world, capacitors and inductors are constant impedances

$$Z_R = R \qquad Z_L = sL \qquad Z_C = \frac{1}{sC}$$

Live in e^st world if we assure
1) Vin(t) is a linear combination of e^st
2) Steady state, e^-t/tau decayed

Easy to achieve if e^st is periodic

Euler's Formula

complex

periodic

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$$

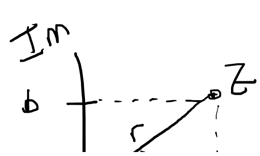
Note

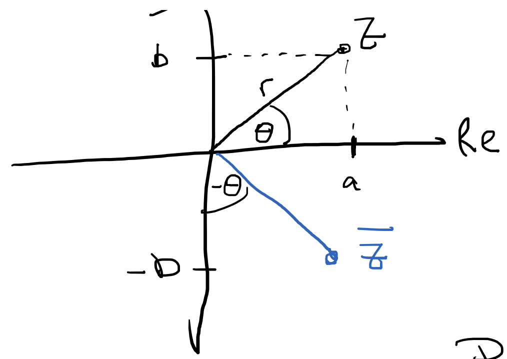
$$\boxed{j = i}$$

Today:
- I. Complex Numbers and Euler's Formula
  - a. Complex Numbers
  - b. Proof of Euler's
  - c. Phasors and e^jwt
  - d. Linear Combinations of e^jwt
    - i. Sinusoids
    - ii. Phasors and Sinusoids
    - iii. Multi-Frequency Combinations
- II. Phasor Domain
  - a. Imaginary Impedance
  - b. Real and Imaginary Impedance -> Transfer Function
  - c. ~~Applications: Power Grid~~

I. Complex Numbers and Euler's Formula

a. Navigate complex numbers





Cartesian

$z = a + jb$

$Re\{z\} = a = r\cos(\theta)$

$Im\{z\} = b = r\sin(\theta)$

Polar

$0 \le \theta < 2\pi$

$\theta = \measuredangle z = \arctan\left(\frac{b}{a}\right)$

$r = |z| = \sqrt{a^2 + b^2}$

real & positive

$|z| = \sqrt{z \cdot \bar{z}}$

$\bar{z}$ conjugate

Complex Conjugate $\bar{z}$

$$\boxed{\overline{(a+jb)} = (a - jb)}$$

$$\sqrt{z \cdot \bar{z}} = \sqrt{(a+jb)(a-jb)}$$

$$= \sqrt{a^2 + jab - jab - (jb)^2}$$

$$= \sqrt{a^2 + b^2}$$

Note

$j = \sqrt{-1} = j$

$j^2 = (\sqrt{-1})^2 = -1$

$j^3 = j j^2 = -j$

$j^4 = j^2 j^2 = 1$

$j^5 = j j^2 j = j$

$j^{-1} = \frac{1}{j}\frac{(j)}{(j)} = \frac{j}{j^2} = -j$

b. Proof of Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Taylor Expansions

$$\cos x = x^0 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{2n!}$$

$$\sin x = x^1 - \frac{x^3}{3!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$e^x = x^0 + x^1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^m}{m!}$$

$$e^{j\theta} = \; = \left(1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \cdots + \frac{(j\theta)^{2n}}{2n!} + \frac{(j\theta)^{2n+1}}{(2n+1)!} + \cdots\right)$$

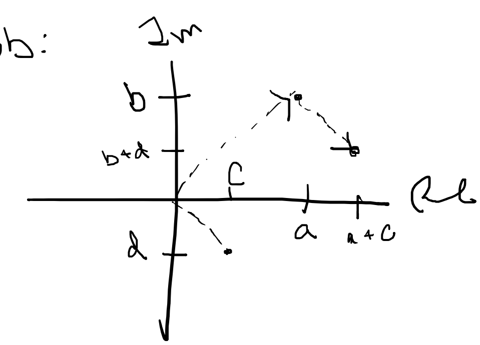
$$= \left(1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \cdots + (-1)^n \frac{\theta^{2n}}{2n!} + j(-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \cdots\right)$$

$$= \left[\left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4} - \cdots + (-1)^n \frac{\theta^{2n}}{2n!} + \cdots\right) + j\left(\theta - \frac{\theta^3}{3} + \cdots + (-1)^n \frac{\theta^{2n+1}}{(2n+1)!} + \cdots\right)\right]$$

$$e^{j\theta} = \cos\theta + j\sin(\theta)$$

c. Navigating the complex plane

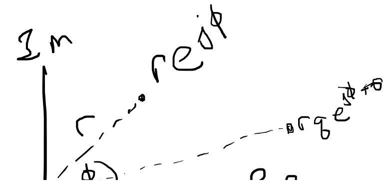
Add/sub:



$$(a + jb) + (c + jd)$$

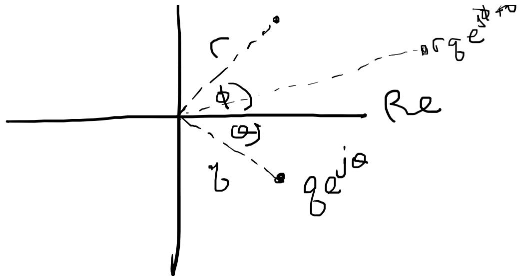
$$(a + c) + j(b + d)$$

Multiply/Divide



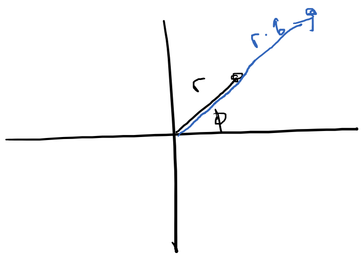
$$re^{j\phi} \cdot qe^{j\theta}$$

$$(r \cdot q)\, e^{j(\phi + \theta)}$$

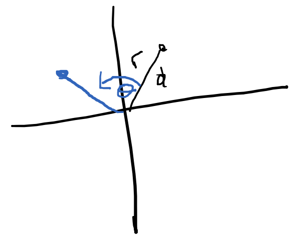


$$(r\cdot q)\, e^{j(\phi+\theta)}$$

Scale $\qquad re^{j\phi}\cdot qe^{j\theta}$



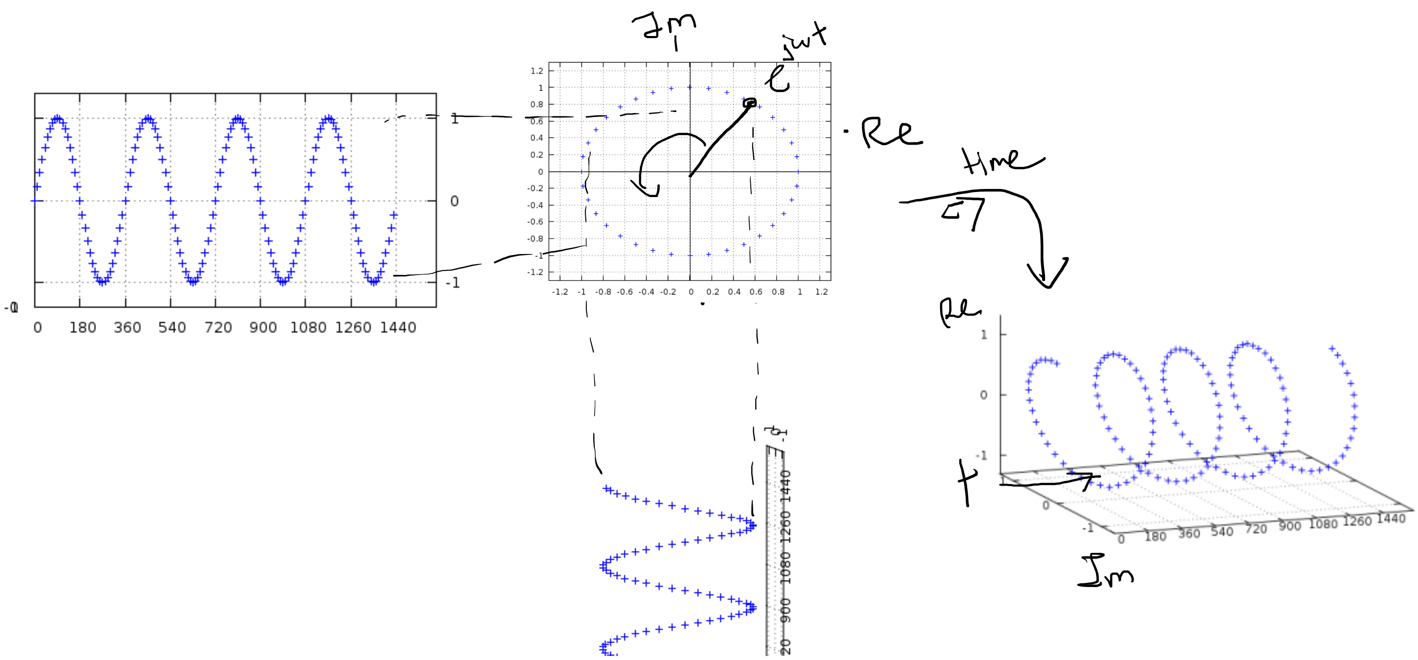
Rotate $\qquad re^{j\phi}\cdot 1e^{j\theta}$

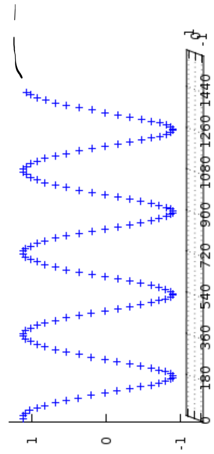


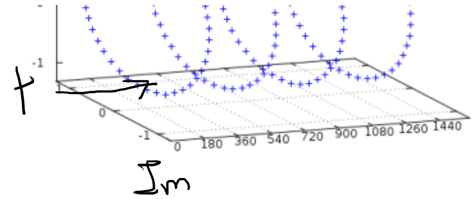
Phasor: $re^{j\phi}$ Contains magnitude and phase information

d. Phasors and Time

$$e^{j\omega t} = \underbrace{\cos(\omega t)}_{Re} + j\underbrace{\sin(\omega t)}_{Im}$$







ω angular frequency $\qquad f$ freq

ω in (rad/sec) $\Rightarrow$ f in Hz = 1/sec

$$\boxed{\omega = 2\pi f}$$

Some interesting frequencies

$\omega_{wifi} = 2\pi f_{wifi}$ $\qquad f_{wifi} = 2.4 \times 10^9$ Hz

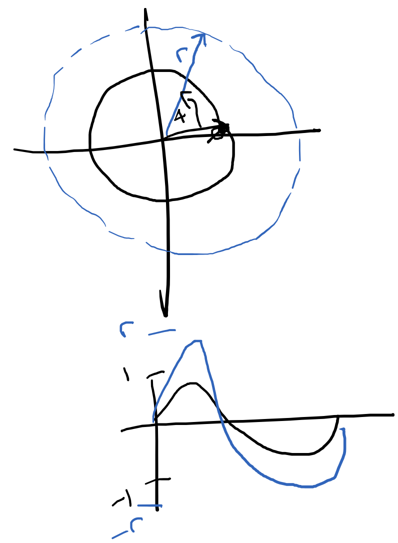
$\omega_{wall} = 2\pi f_{wallpower}$ $\qquad f_{wallpower} = 60$ Hz

$\omega_{human} = 2\pi f_{HH}$ $\qquad f_{\substack{human\\hearing}} = \sim 20 - 20 \times 10^3$ Hz

<u>scaling</u> $\underline{e^{j\omega t}}$ $\qquad$ <u>rotating</u> $\underline{e^{j\omega t}}$

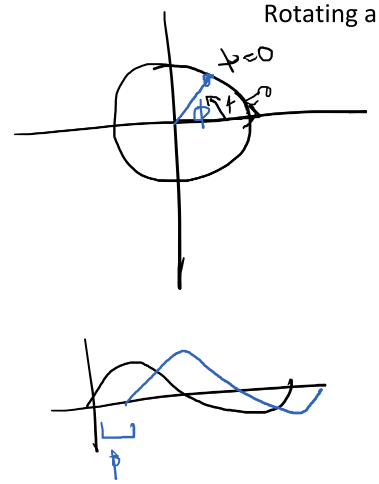
$me^{j0} \cdot e^{j\omega t}$ $\qquad 1e^{j\theta} \cdot e^{j\omega t}$

$re^{j0} \cdot e^{j\omega t}$



$1e^{j\phi} \cdot e^{j\omega t}$

Rotating a clock



E. Linear Combinations of e^jwt

i. Sinusoids

Note

$\cos(-\theta) = \cos(\theta)$

$\sin(-\theta) = -\sin(\theta)$

$$e^{j\omega t} + \overline{e^{j\omega t}}$$

$$e^{j\omega t} + e^{-j\omega t} = \left[\cos(\omega t) + \cancel{j\sin(\omega t)}\right] + \left[\cos(-\omega t) + \cancel{j\sin(-\omega t)}\right]$$

$$= 2\cos(\omega t)$$

$$\boxed{\begin{aligned}\cos(\omega t) &= \frac{1}{2}\left(e^{j\omega t} + \overline{e^{j\omega t}}\right)\\ \sin(\omega t) &= \frac{1}{2j}\left(e^{j\omega t} - \overline{e^{j\omega t}}\right)\end{aligned}}$$

ii. Phasors sinusoids

$$r\cos(\omega t + \phi) = \frac{1}{2}\left(re^{j\omega t + \phi} + \overline{re^{j\omega t + \phi}}\right)$$

$$r\cos(\omega t + \phi) = \frac{1}{2}\left(re^{j\omega t + \phi} + \overline{re^{j\omega t + \phi}}\right)$$

Fast and loose math fixed ->

$$= \frac{1}{2}\left(\underbrace{re^{j\phi}}_{\text{phasor}} \underbrace{e^{j\omega t}} + \underbrace{\overline{re^{j\phi}}} \cdot \underbrace{\overline{e^{j\omega t}}}\right)$$

$\cos(\omega t)$

iii. Multi- frequency combinations

$$re^{j(\omega_1 t + \phi)} + re^{j(\omega_2 t + \phi)} = re^{j\phi} \cdot \left(e^{j\omega_1 t} + e^{j\omega_2 t}\right)$$

$$\underbrace{re^{j\phi}}\left(\text{linear combination } e^{j\omega t_s}\right)$$

II. Phasor Domain

$$V_{in}(t) = \underbrace{\tilde{V}_{in}}_{\text{phasor}} e^{st} \xrightarrow{\text{now}} V_{in}(t) = re^{j(\omega t + \phi)} = \underbrace{re^{j\phi}}_{\text{phasor}} \cdot e^{j\omega t}$$

phasor

scale & mag. manipulation

$$\underbrace{\tilde{V}_{in}e^{j\omega t}} \rightarrow \boxed{\text{circuit}} \rightarrow \underbrace{\lambda \tilde{V}_{in}} e^{j\omega t}$$

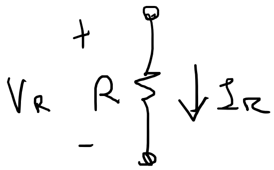
Phasor Domain

$$\boxed{Z_R = R \qquad Z_L = j\omega L \qquad Z_C = \frac{1}{j\omega C}}$$

What is an imaginary impedance?

Real Impedance

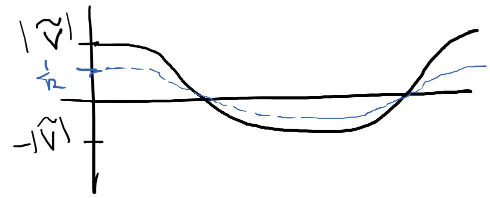
Real Impedance



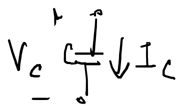
$V_R = R\, I_R$

$V_R = \underbrace{\tilde{V}_R}_{\text{phasor}} \cos(\omega t)$

$Z_R = \frac{\tilde{V}}{\tilde{I}} = R \quad \tilde{I} = \frac{\tilde{V}}{R}$



a. Imaginary Impedance

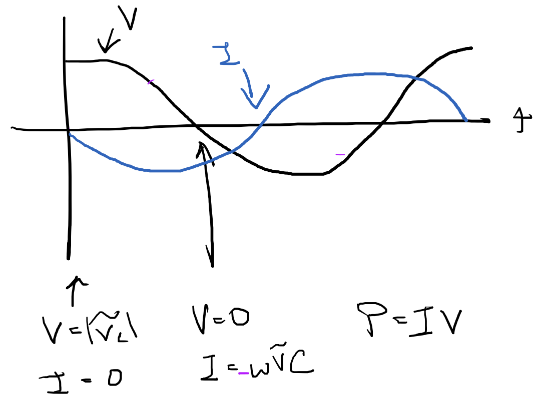


$V_c = \tilde{V}_c \cos(\omega t)$

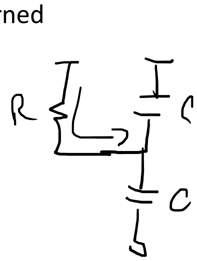
$I_c = \frac{d}{dt} V_c \cdot C$

$I_c = \frac{d}{dt} (\tilde{V}_c \cos(\omega t)) \cdot C$

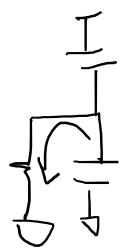
$I_c = \omega(-\sin(\omega t))\tilde{V} \cdot C$



Any power through a purely real impedance is used
Any power through a purely imaginary impedance is returned

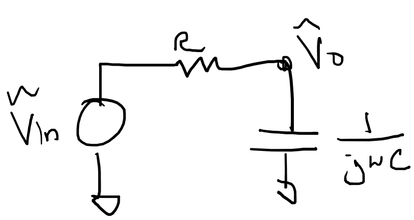


$\rightarrow$



Capacitor returns power stored in it

b. Real and Im Impedance -> Transfer Function



Manipulating Phasor (Defined by circuit H(jw))

Chosen input

$\tilde{V}_o = \underbrace{\Pi}_{\text{Manipulating Phasor}}\ \underbrace{\tilde{V}_{in}}_{\text{Chosen input}}$

$V_{in} = \tilde{V}_{in}(\text{linear combo } e^{st})$

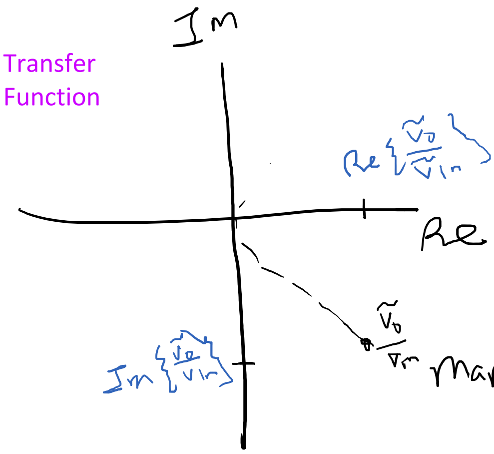
$\tilde{V}_o = \tilde{V}_{in} \underbrace{\frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R}}$

Manipulating phasor : H(jw)

$$\underbrace{\frac{\tilde{V}_o}{\tilde{V}_{in}} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}}_{H(j\omega)} = \frac{(1 - j\omega RC)}{(1 - j\omega RC)} \cdot \frac{1}{1 + j\omega RC} = \frac{1 - j\omega RC}{1 + (\omega RC)^2}$$

$$Re\left\{\frac{\tilde{V}_o}{\tilde{V}_{in}}\right\} = \frac{1}{1 + (\omega RC)^2} \qquad Im\left\{\frac{\tilde{V}_o}{\tilde{V}_{in}}\right\} = \frac{-j\omega RC}{1 + (\omega RC)^2}$$

Transfer Function

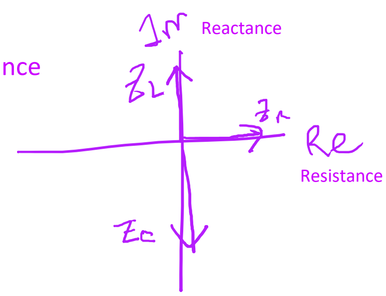


manipulating phasor: $|H(j\omega)|\, e^{j\angle H(j\omega)}$

Transfer Function

$$\boxed{\frac{\tilde{V}_o}{\tilde{V}_{in}} = H(j\omega)}$$

Passive Impedance



$$Z_R = R$$

$$Z_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C}$$

$$Z_L = j\omega L$$