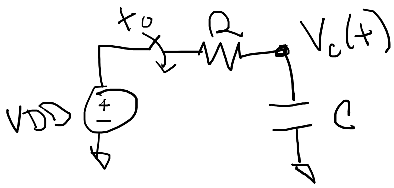
# 6-25, Lecture 4: Time-Varying Complex Exponential Inputs

Thursday, June 25, 2020 10:24 AM

## Last time on EE16B...

(Post lecture notes in purple)
(Impt equations boxed in green)

Developed a math model

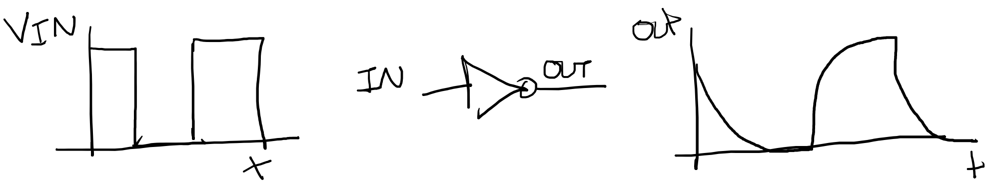



$$\frac{d}{dt}V_c(t) + \frac{1}{RC}V_c(t) = \frac{VDD}{RC}$$

Diff eq:

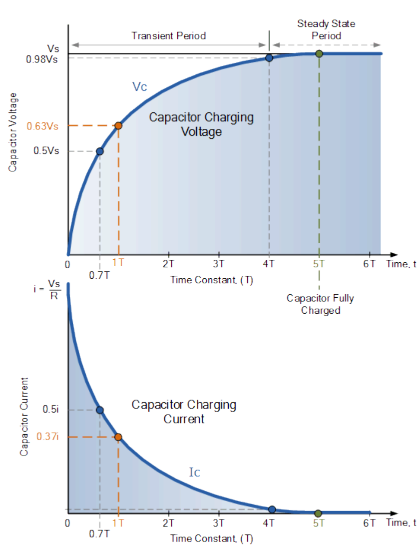
$$\frac{d}{dt}x(t) + ax(t) = b \Rightarrow x(t) = b/a + (x_0 - b/a)e^{-at}$$

Model a sq wave through an inv




Time constant $\tau = RC$

$$V_c(t) = V_{DD}(1 - e^{-t/RC})$$




| Time Constant | RC Value | Percentage of Maximum Voltage |
|---|---|---|
| 0.5 time constant | 0.5T = 0.5RC | 39.3% |
| 0.7 time constant | 0.7T = 0.7RC | 50.3% |
| 1.0 time constant | 1T = 1RC | 63.2% |
| 2.0 time constants | 2T = 2RC | 86.5% |
| 3.0 time constants | 3T = 3RC | 95.0% |
| 4.0 time constants | 4T = 4RC | 98.2% |
| 5.0 time constants | 5T = 5RC | 99.3% |

Looked at
real world applications
(ADCs)

Today –

I. Non-constant inputs to RC

A. Diff Eqn ⇒ Model

B. Check w/ constant

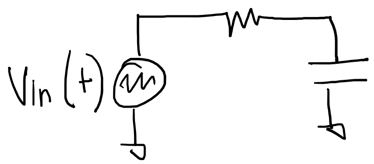
C. Exp Inputs

i. L & C Impedance

ii. Steady State

iii. Euler's Function

Goal I




$V_{in}(t)$ = any time varying function $f(t)$

Prev: $\frac{d}{dt}V_o(t) + \frac{1}{RC}V_o(t) = \frac{V_{inc}}{RC}$

Now: " " " " $= \frac{V_{in}(t)}{RC}$

* Math Break *

Goal A. Solve

$$\frac{d}{dt}x(t) + a x(t) = g(t)$$

Last Time

soln $x(t) = (x_0 - b/a)\underbrace{e^{-at}}_{\text{homogenous part}} + b/a$

$\Rightarrow x(t) = x_0 e^{-at}$

This time

$x(t) = x_g(t) + x_p(t)$

Several ways to solve eqn:

A) Piecewise Approx

Fa19 Note "Input"

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Fa19 Note "Input"
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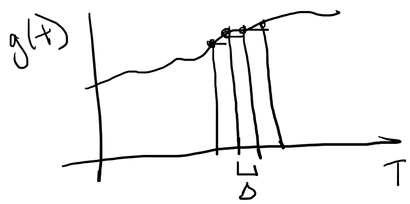



$\Delta \to 0$

B) Integrating Factor

Consider <u>product rule</u>

$$\frac{d}{dt}\left(x(t)y(t)\right) = \frac{dy(t)}{dt}x(t) + \frac{dx(t)}{dt}y(t)$$

$$\left[\frac{d}{dt}x(t)\right]y(t) + \left[\frac{d}{dt}y(t)\right]x(t)$$

$$\frac{d}{dt}x(t)\,y(t) + a\,y(t)\,x(t) = g(t)y(t)$$

Choose $y(t)$ s.t. $\frac{d}{dt}y(t) = ay(t) \Rightarrow y(t) = e^{at}$

$$\underbrace{\frac{d}{dt}x(t)y(t) + \frac{d}{dt}y(t)\,x(t)}_{\text{product rule}} = g(t)y(t)$$

$$\int \frac{d}{dt}\left(x(t)y(t)\right) = \int g(t)y(t)$$

$$x(t)y(t) = \int_{t_0}^{t} g(\tau)y(\tau)d\tau + C$$

$$x(t)e^{at} = \int_{t_0}^{t} g(\tau)e^{a\tau}d\tau + C$$

Contains ✓ homogeneous part

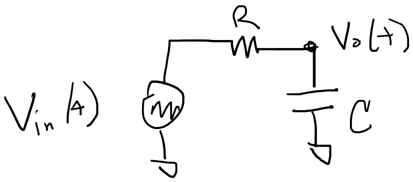
$$\boxed{x(t) = e^{-at}\int_{t_0}^{t} g(\tau)e^{a\tau}d\tau + Ce^{-at}}$$

$$x(0) = x_0 = \int_0^0 g(\tau)e^{a\tau}d\tau + C$$

$$\boxed{C = x_0}$$

$$\boxed{C = X_0}$$

*Math Break Over*




$V_{in}(t)$ see how an RC circuit affects

Circuit

$$\frac{d}{dt} V_o(t) + \frac{1}{RC} V_o(t) = \frac{V_{in}(t)}{RC}$$

$$\boxed{V_o(t) = e^{-t/RC} \int_{t_0}^{t} \frac{V_{in}(\theta)}{RC} e^{\theta/RC} \, d\theta + V_o(0) e^{-t/RC}}$$

$(\theta \to \tau \quad \cancel{K \to C})$

soln RC circuit

general

$$\frac{d}{dt} x(t) + a x(t) = g(t)$$

$$x(t) = e^{-at} \int_{t_0}^{t} g(\tau) e^{a\tau} \, d\tau + X_0 e^{-at}$$

Subgoal B. Check using a const

$V_{in}(t) = VDD \qquad V_o(0) = 0 \qquad g(t) = \frac{VDD}{RC}$

$$V_o(t) = e^{-t/RC} \int_{t_0}^{t} \frac{VDD}{RC} e^{\theta/RC} \, d\theta + V_o(0) e^{-t/RC}$$

$$= e^{-t/RC} \frac{VDD}{\cancel{RC}} \cdot \cancel{RC} \left[ e^{t/RC} - e^{0/RC} \right] + 0 \, e^{-t/RC}$$

$$= VDD \left( e^{(t-t)/RC} - e^{(0-t)/RC} \right)$$

$$V_o(t) = VDD - VDD e^{-t/RC}$$

Matches Yesterday!

## Sin

$V_{in}(t) = \sin(t) \qquad V_o(0) = X_0$

$$V_o(t) = e^{-t/RC} \underbrace{\int_{t_0}^{t} \frac{\sin(\theta)}{RC} e^{\theta/RC}\, d\theta}_{\text{hard intergal}} + V_o(0) e^{-t/RC}$$

## Subgoal C. Exp Inputs

Nice things about $e^{st}$ ($s$ is some const)

- eigenfunction
- $e^{s\theta} \cdot e^{\theta/RC} = e^{\theta(s + \frac{1}{RC})}$ easy to intg
- Emily's ~~second~~ favorite function

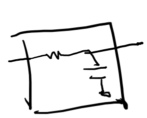
### Try it

$V_{in}(t) = \tilde{V} e^{st}$ , $\tilde{V}$ some const scalar & $V_o(t_0) = V_o(0)$ as initial condition

$$V_o(t) = e^{-t/RC} \int_0^{t} \frac{\tilde{V}}{RC} e^{\theta(s + \frac{1}{RC})} + V_o(0) e^{-t/RC}$$

$$= e^{-t/RC} \left[ e^{t(s + \frac{1}{RC})} - e^{0} \right] \frac{\tilde{V}}{RC} \frac{1}{s + \frac{1}{RC}} + V_o(0) e^{-t/RC}$$

$$V_o(t) = \frac{\tilde{V}}{sRC + 1} e^{st} \qquad + \left( V_o(0) - \frac{\tilde{V}}{sRC+1} \right) e^{-t/RC}$$

$\tilde{V} e^{st} \rightarrow$  $\rightarrow \lambda \tilde{V} e^{st}$

True if we could get rid of $e^{-t/RC}$ term

Choose $V_o(0)$ s.t. $\left(V_o(0) - \frac{\tilde{V}}{sRC+1}\right)e^{-t/RC} = 0$

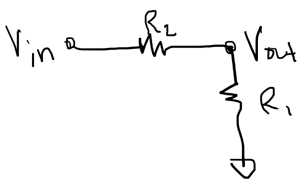
$$V_o(0) = \frac{\tilde{V}}{sRC+1}$$

$$\boxed{V_o(t) = \frac{1}{sRC+1}\tilde{V}e^{st}}$$

RC circuit for convenient initial condition

Subsubgoal i. L & C why we care about exp inputs.

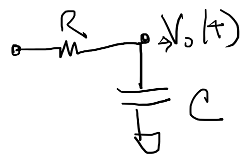
Voltage Divider




$$V_{out} = V_{in}\frac{R_1}{R_1+R_2}$$

Eigenfunction property

$V_{in}(t) = \tilde{V}e^{st} \rightarrow$



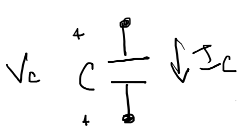

$$V_o(t) = \tilde{V}e^{st}\left(\frac{\frac{1}{sC}}{R + \frac{1}{sC}}\right)$$

$e^{st}$ world

scalars/magnitudes

$$V = \tilde{V}e^{st} \qquad I = \tilde{I}e^{st}$$

Capacitor



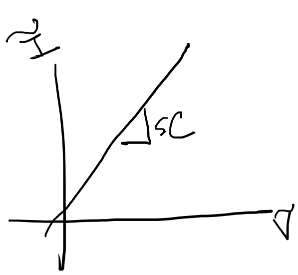

$$I_c = \frac{d}{dt}V_c \cdot C$$

$$\tilde{I}e^{st} = \frac{d}{dt}\tilde{V}e^{st} \cdot C$$

$$\tilde{I}e^{st} = s\tilde{V}e^{st} \cdot C$$

$$\frac{\tilde{V}e^{st}}{\tilde{I}e^{st}} = \frac{1}{sC}$$

$$\boxed{\frac{\tilde{V}}{\tilde{I}} = \frac{1}{sC}}$$

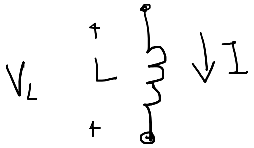



Aside

$$\frac{V}{I} = R$$

Inductor

Inductor

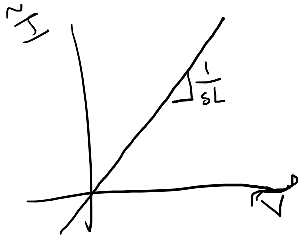



$$V_L = \tilde{V}_L e^{st} \qquad I_L = \tilde{I}_L e^{st}$$

$$V_L = \frac{d}{dt} I_L \cdot L$$

$e^{st}$ world

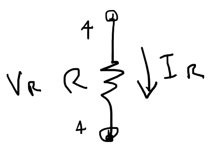
$$\tilde{V}_L e^{st} = \frac{d}{dt} \tilde{I}_L e^{st} \cdot L$$




$$\tilde{V}_L e^{st} = s \tilde{I}_L e^{st} \cdot L$$

$$\boxed{\frac{\tilde{V}_L e^{st}}{\tilde{I}_L e^{st}} = sL}$$

Resistor

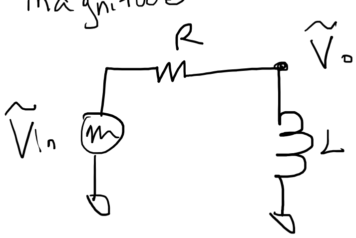



$$V_R = \tilde{V}_R e^{st} \qquad I_R = \tilde{I}_R e^{st}$$

$$\tilde{V}_R e^{st} = \tilde{I} e^{st} R$$

$$\boxed{\frac{\tilde{V}_R e^{st}}{\tilde{I}_R e^{st}} = R}$$

magnitude divider




$\tilde{V}_x$ - scalar / magnitude

$V_x = \tilde{V}_x e^{st}$ final answer

$$\tilde{V}_R = \tilde{I}_R R = \tilde{V}_{in} - \tilde{V}_o$$

$$\tilde{V}_L = \tilde{I}_L sL = \tilde{V}_o - 0$$

$$\tilde{I}_R = \tilde{I}_L$$

$$\frac{\tilde{V}_{in} - \tilde{V}_o}{R} = \frac{\tilde{V}_o}{sL}$$

$$sL(\tilde{V}_{in} - \tilde{V}_o) = R\tilde{V}_o$$

$$sL\tilde{V}_{in} = sL\tilde{V}_o + R\tilde{V}_o$$

$$\tilde{V}_o = \tilde{V}_{in} \frac{sL}{\phantom{sL+R}}$$

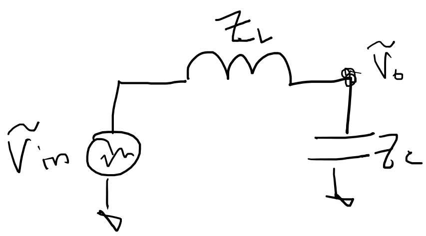
$$sL + R$$

$$\underbrace{\tilde{V}_o e^{st}} = \tilde{V}_{in} e^{st} \frac{sL}{sL+R}$$

$$V_o = \tilde{V} e^{st} \frac{sL}{sL+R}$$

$$\boxed{\begin{aligned} Z_L &= sL \\ Z_C &= \frac{1}{sC} \\ Z_R &= R \end{aligned}}$$

More complicated




$$\tilde{V}_o = \tilde{V}_{in} \frac{Z_C}{Z_C + Z_L}$$

$$= \tilde{V}_{in} \frac{\frac{1}{sC}}{\frac{1}{sC} + sL}$$

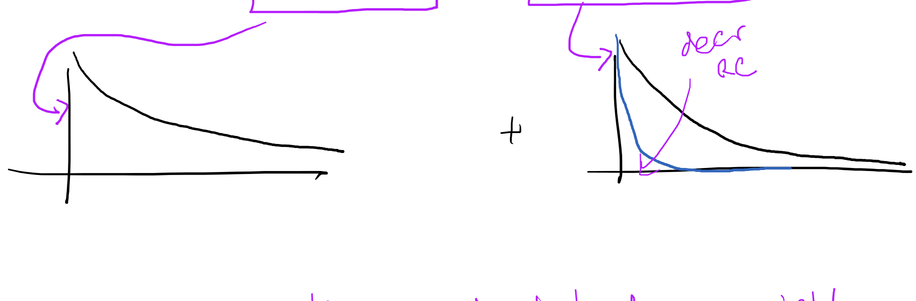
Want to be in $e^{st}$ world for L & C

Subsubgoal ii. Steady state

$V_o(0) \neq \frac{\tilde{V}}{sRC+1}$ What happens?

Some small scalar

$$V_o(t) = \underbrace{\frac{\tilde{V}}{sRC+1} e^{st}} + \underbrace{\varepsilon_0 e^{-t/RC}}$$




choose RC time constant to decay quickly

Would be nice if we could

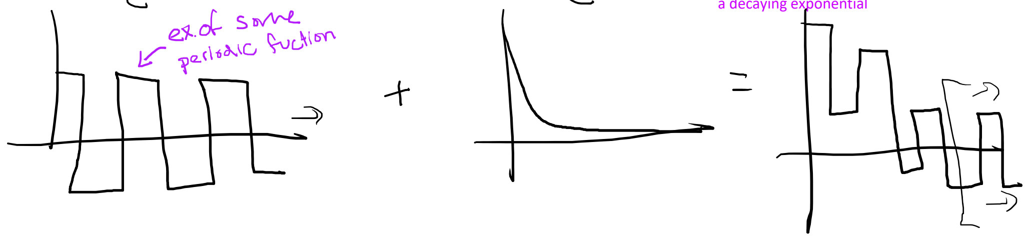
Choose some s s.t. $e^{st}$ was periodic

$-t/RC$ ← The size of the RC time constant doesn't matter if s makes e^st periodic, because a

Would be nice if we could

Choose some $s$ s.t. $e^{st}$ was periodic

The size of the RC time constant doesn't matter if s makes e^st periodic, because a periodic function will always outlast a a decaying exponential




## Steady state:

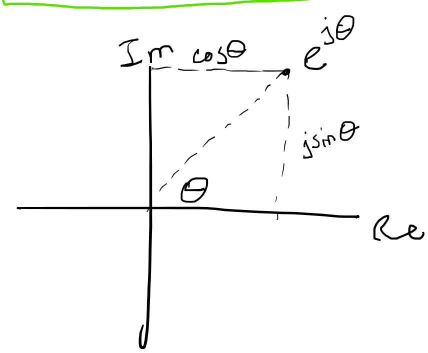
Wait till $e^{-t/RC}$ is negligible & $K$ is effectively 0

subsubgoal. iii

Euler's Formula

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta} \qquad (j = i = \sqrt{-1})$$

Cartesian & polar like →




$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$

$$\sin\theta = \frac{1}{2j}\left(e^{j} - e^{-j\theta}\right)$$