Bonus Lecture

Quadratic Programming

8/12/20

Today: A very useful concept in control / optimization

$$\boxed{\text{Quadratic Programming}}$$

We have already seen two types of optimization problems

$$\min_x \|x\| \quad \text{s.t.} \quad Ax = b$$

A full row-rank

$$\min_x \|Ax - y\|$$

A full column-rank

These are both special cases of convex quadratic programs

QP (Quadratic Program)

Equality - Constrained case

$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{2} x^T Q x + x^T q + c$$

$$\text{s.t.} \quad Ax = b$$

$$f(x) = \frac{1}{2} x^T Q x + x^T q + c$$

$$Q \in \mathbb{R}^{n \times n} \qquad q \in \mathbb{R}^n \qquad c \in \mathbb{R}$$

Quadratic function

We can always assume that

Q is symmetric.

$$x^T Q x = x^T \left(\frac{1}{2}(Q + Q^T) + \frac{1}{2}(Q - Q^T) \right) x$$

$$= x^T \left(\frac{Q + Q^T}{2} \right) x + x^T \left(\frac{Q - Q^T}{2} \right) x$$

$$+ \frac{x^T Q x}{2} - \frac{x^T Q^T x}{2}$$

$$= x^T \left(\frac{Q + Q^T}{2} \right) x$$

If Q is not symmetric

define $\hat{Q} = \dfrac{Q + Q^T}{2}$

$$\frac{1}{2} x^T Q x + x^T q + c = \underline{\underline{\frac{1}{2} x^T \hat{Q} x + x^T q + c}}$$

$$f(x) = \frac{1}{2} x^T Q x + x^T q + c$$

What is the minimum x^*?

Does $f(x)$ have a minimum?

e.g. $f(x) = -\frac{1}{2} x^2$

e.g. $f(x) = 0 \cdot x^2 + x$?

$f(x)$ $a x^2 + b x + c$

$a > 0 \Rightarrow f(x)$ has a minimum

Assumption:

Q has strictly positive eigenvalues (Positive Definite)

Look at Q:

$$QV = V\Lambda \implies Q = V\Lambda V^{-1}$$

$$Q = Q^T = V^{-T}\Lambda V^T$$

$$V^{-1} = V^T$$

$$Q = V\Lambda V^T$$

Define:

$$Q^{1/2} := \Lambda^{1/2} V^T$$

$$\Lambda^{1/2} := \operatorname{diag}\left(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}\right)$$

$$Q^{-1/2} := V\Lambda^{-1/2}$$

$$\Lambda^{-1/2} := \operatorname{diag}\left(\frac{1}{\sqrt{\lambda_1}}, \frac{1}{\sqrt{\lambda_2}}, \dots, \frac{1}{\sqrt{\lambda_n}}\right)$$

$$z := Q^{1/2} x$$

$$x := Q^{-1/2} z$$

$$f(x) = \frac{1}{2} x^T Q x + x^T q + c$$

$$= \frac{1}{2} x^T Q^{1/2^T} Q^{1/2} x + x^T q + c$$

$$= \frac{1}{2} z^T z + z^T \underbrace{Q^{-1/2^T} q}_{\tilde{q}} + c$$

$\tilde{q} = Q^{-1/2^T} q$

$$= \frac{1}{2} \|z\|_2^2 + z^T \tilde{q} + c$$

$$= \frac{1}{2}\left(z_1^2 + \dots + z_n^2\right) + z_1 \cdot \tilde{q}_1 + \dots + z_n \cdot \tilde{q}_n + c$$

$$= c + \sum_{i=1}^{n} \underbrace{\left(\frac{1}{2} z_i^2 + \tilde{q}_i \cdot z_i\right)}$$

Minimize z_i independently

Minimum is attained when $z_i^* = -\tilde{q}_i$

$$z^* = -\tilde{q}$$

$$Q^{1/2} x^* = -Q^{-1/2^T} q$$

$$Q^{-1/2} Q^{1/2} x^* = -Q^{-1/2} Q^{-1/2^T} q$$

$$x^* = -V \Lambda^{-1/2} \Lambda^{-1/2} V^T q$$

$$= -V \Lambda^{-1} V^T q$$

$$\boxed{x^* = -Q^{-1} q}$$

$$\nabla_x f(x) = \nabla_x \left(\underbrace{\frac{1}{2} x^T Q x} + \underline{q}^T x + c \right)$$

$$= Qx + q$$

(x^* is where gradient of $F(x)$ is 0)

Lets return to our QP

$$\boxed{\min_{x} \quad \frac{1}{2} x^T Q x + x^T q + c \qquad \text{s.t.} \quad Ax = b}$$

e.g.
$$\min_{x_1, x_2} \quad 2x_1^2 + 3x_1 - x_2^2$$
$$\text{s.t.} \quad x_2 = 1$$

Assumption: $b \in \text{Col}(A)$

"Problem is feasible / solution exists"

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

$A \in \mathbb{R}^{m \times n}$

Define $x \in \mathbb{R}^n$

$$x := V_1 z_1 + V_2 z_2$$

$z_1 \in \mathbb{R}^r$ $\qquad$ $z_2 \in \mathbb{R}^{n-r}$

$$A(V_1 z_1^* + V_2 z_2^*) = b$$

$$[U_1 \;\; U_2]\begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}(V_1 z_1^* + \underline{V_2} z_2^*) = b$$

$$[U_1 \;\; U_2]\begin{bmatrix} S \\ 0 \end{bmatrix} \cancel{V_1^T V_1} z_1^* = b$$

$$[U_1 \;\; U_2]\begin{bmatrix} S z_1^* \\ 0 \end{bmatrix} = b$$

$$\begin{bmatrix} S z_1^* \\ 0 \end{bmatrix} = \begin{matrix} U_1^T b \\ U_2^T b \end{matrix}$$

$$z_1^* = S^{-1} U_1^T b$$

$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{2} x^T Q x + x^T q + c$$

$$\text{s.t.} \quad Ax = b$$

$$\Updownarrow$$

$$\min_{z_2 \in \mathbb{R}^{n-r}} \quad \frac{1}{2}(V_1 z_1^* + V_2 z_2)^T Q (V_1 z_1^* + V_2 z_2) + (V_1 z_1^* + V_2 z_2)^T q + c$$

$$\min_{z_2 \in \mathbb{R}^{n-r}} = \frac{1}{2} z_2^T \underbrace{\left(V_2^T Q V_2\right)} z_2 + z_2^T \left(V_2^T \left(Q V_1 z_1^* + q\right)\right) + k$$

We require that $V_2^T Q V_2$

is Positive Definite

(positive eigenvalues)

$$z_2^* = -\left(V_2^T Q V_2\right)^{-1} \left(V_2^T \left(Q \underbrace{V_1 S^{-1} U_1^T} b + q\right)\right)$$

$$= -\left(V_2^T Q V_2\right)^{-1} \left(V_2^T \left(Q A^+ b + q\right)\right)$$

$$x^* = \underbrace{V_1 S^{-1} U_1^T b} - V_2 \left(V_2^T Q V_2\right)^{-1} V_2^T \left(Q A^+ b + q\right)$$

$$= A^+ b - V_2 \left(V_2^T Q V_2\right)^{-1} V_2^T \left(Q A^+ b + q\right)$$

If QP is constrained

else: $x^* = -Q^{-1} q$

x^* given on prev. page

is solution to

$$\min_{x \in \mathbb{R}^n} \quad \frac{1}{2} x^T Q x + x^T q + c$$

$$\text{s.t.} \quad Ax = b$$

If: $b \in \text{col}(A)$

and $V_2^T Q V_2 \succ 0$ positive definite

Return to least-squares problems:

$$\min_x \|Ax - b\|_2 = \min_x \frac{1}{2} \|Ax - b\|_2^2$$

$$= \min_x \frac{1}{2} x^T A^T A x - x^T A b + \frac{b^T b}{2}$$

$$Q = \boxed{A^T A}$$

$$q = -Ab$$

$$\min_x \|x\| \quad \text{s.t. } Ax = b \quad = \quad \min_x \frac{1}{2} x^T x \quad \text{s.t. } Ax = b$$

$$Q = I$$

$$q = 0$$

Now:

LQR

"Linear - Quadratic Regulator"

$$\min_{x_0, u_0, \ldots, x_T} \sum_{t=0}^{T-1} \left(\frac{1}{2} x_t^T Q_t x_t + x_t^T q_t + \frac{1}{2} u_t^T R_t u_t + u_t^T r_t \right) + x_T^T Q_T x_T$$

$$x_0 = \hat{x}$$

$$x_1 = A x_0 + B u_0 + c$$

$$\vdots$$

$$x_T = A x_{T-1} + B u_{T-1} + c$$

define $$z := \begin{bmatrix} x_0 \\ u_0 \\ x_1 \\ u_1 \\ \vdots \\ \\ x_T \end{bmatrix}$$

$$\min_{z} \quad \frac{1}{2} \quad z^T \begin{bmatrix} Q_0 & & & & & \\ & R_0 & & & & \\ & & Q_1 & & & \\ & & & R_1 & & \\ & & & & \ddots & \\ & & & & & Q_T \end{bmatrix} z \quad + \quad z^T \begin{bmatrix} q_0 \\ r_0 \\ q_1 \\ r_1 \\ \vdots \\ q_T \end{bmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} I & & & & & & \\ -A & -B & I & & & & \\ & & -A & -B & I & & \\ & & & & \ddots & & \\ & & & & -A & -B & I \end{bmatrix} z = \begin{bmatrix} \hat{x} \\ c \\ c \\ \vdots \\ c \end{bmatrix}$$