8-10, Lecture 29: Sampling and Aliasing

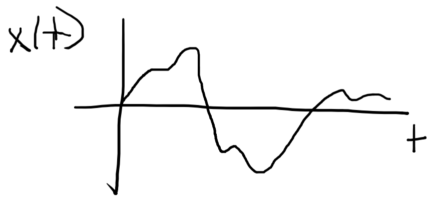
Monday, August 10, 2020 9:32 AM

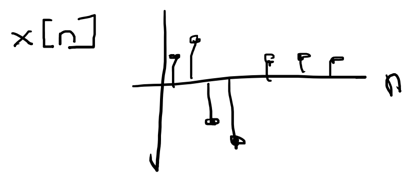
(Post lecture notes in purple)
(Impt equations boxed in green)

I. ADCs
II. Discrete Frequency Domain
    a. Intuitive Review of DFT Matrix
    b. Sampling Frequency and the DFT
    c. Sampling and Aliasing Examples
III. Sampling
    a. Nyquist Theorem
    b. Sampling Window
        i. Application Example
IV. Anti-Aliasing Filter

Sampling and Aliasing

The world is continuous -however- Computers need discrete





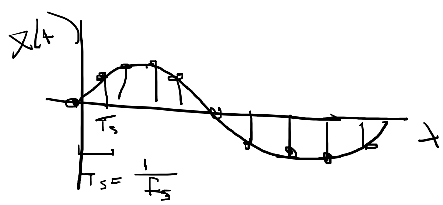
I. Analog to Digital Converter (ADC)

Given a cont. time waveform and an ADC that samples at freq (f_s), collect N samples

Time Domain

Sampling freq: f_s

Sampling Window: N



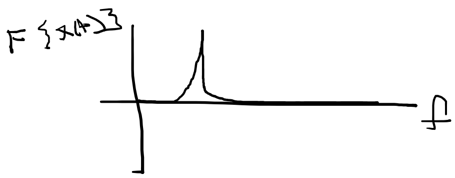
$x[n] =$

$[x(0)\ \ x(T_s)\ \ x(2T_s) \cdots x((N-1)T_s)]$

$\Downarrow$ FT

$\Downarrow$ DFT

Frequency Domain



every freq

$0 \to \infty$

w/ infinite res.

$$F x = \begin{bmatrix} \alpha_0 e^{\phi_0} \\ \alpha_1 e^{\phi_1} \\ \vdots \\ \alpha_{N-1} e^{\phi_{N-1}} \end{bmatrix}$$

limited to discrete freq

II. Intuitive Review of DFT

Purpose of DFT?

- Find how much of each frequency is an a signal (data analysis)
  - -- filter design
  - -- signal processing
- Project the signal into a basis of orthogonal sinusoids (math easier)
  - --phasor domain
  - -- convolution

a. DFT Matrix

$$F x = \begin{bmatrix} \text{---}\ U_0\ \text{---} \\ \text{---}\ U_1\ \text{---} \\ \text{---}\ U_2\ \text{---} \\ \vdots \\ \text{---}\ U_{N-1}\ \text{---} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}$$

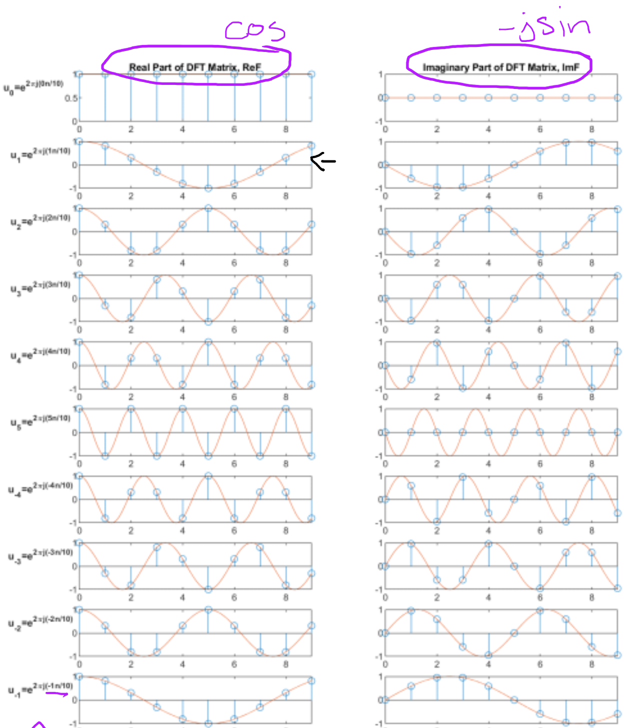
$$U_k = e^{-j\frac{2\pi k}{N} n}$$

↑ traversals around the unit circle @ diff freq

Visual Representation of F for N samples

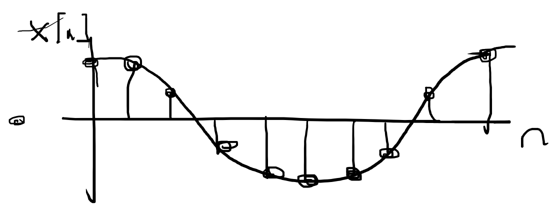
$\frac{1}{\sqrt{N}} \cdot$

(Note there is a scalar here that Matlab ignores. This matrix is not orthonormal, just orthogonal. So I'm applying the scalar out here)



aliasing as negative freq

$$x(4) = \alpha \cos\left(\frac{2\pi 4}{N}\right)$$



$$F x = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \\ \\ \end{array} \begin{bmatrix} 0 \\ \alpha - j0 \\ 0 \\ 0 \\ \vdots \\ \alpha + j0 \end{bmatrix}$$

<- sinusoids are orthogonal

<- aliasing

b. Sampling Frequency and the DFT

How f_s relates to our DFT matrix

$$V_k = e^{\underbrace{-j\frac{2\pi k}{N}}_{\substack{\text{Discrete freq}\\ \omega}} n}$$

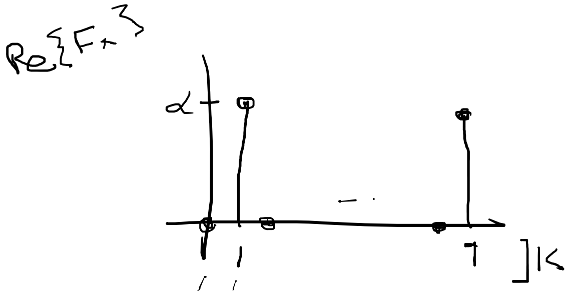
$$\omega = \frac{2\pi k}{N} \quad \text{where} \quad \boxed{\frac{k}{N} = \frac{f}{f_s}}$$

(for pleasant integer ratios of f/f_s)

Draw Prev example

$[0 \ \alpha \ 0 - - \ 0 \ \alpha]$

Re{F_k}



$\frac{0}{10}f_s \quad \frac{1}{10}f_s \quad \cdots \quad \frac{9}{10}f_s$] f (Hz)

$\frac{2\pi 0}{10}f_s \quad \frac{2\pi 1}{10}f_s \quad \cdots \quad \frac{2\pi 9}{10}f_s$] ω (rad)

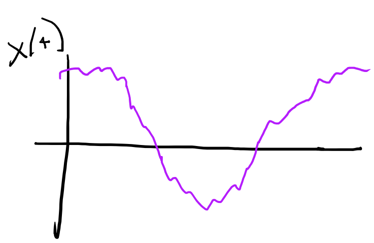
Aliasing

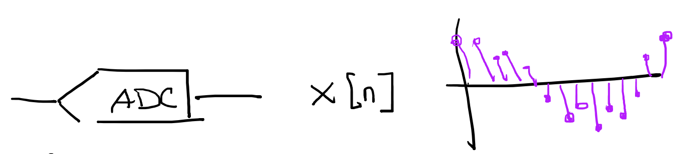
only happens in sampled signals

c. Examples of Sampling and Aliasing

Ex 1 Multi-Tone Signal

$$x_1(t) = \alpha_1 \cos(2\pi \cdot 10Hz \cdot t) + \alpha_2 \sin(2\pi \cdot 40Hz \cdot t)$$





$f_s = 100Hz$

$N = 10$

Convert $f_1 = 10Hz$ & $f_2 = 40Hz$ into k

$f_1 = 10Hz \qquad \frac{k}{N} = \frac{f}{f_s} \Rightarrow \qquad \frac{k}{10} = \frac{10}{100} \Rightarrow 1, \ 10-1$ (from aliasing)

$f_2 = 40Hz \qquad \frac{k}{10} = \frac{40}{100} \Rightarrow 4, \ 10-4$

$f_2 = 40\,Hz$

$\frac{k}{10} = \frac{40}{100} \Rightarrow 4,\ 10-4$

$$Fx = \begin{array}{ccccccc} 0 & 1 & 2 & & & & 9 \\ & & & & & & N-1 \end{array}$$
$$Fx = [\, X \cdot V_0 \quad X \cdot U_1 \quad X \cdot U_2 \quad \cdots\cdots \quad X \cdot U_{N-1} \,]$$

$$\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$
$$= [\ 0 \quad \alpha_1 \quad 0 \quad 0 \quad -j\alpha_2 \quad 0 \quad +j\alpha_2 \quad 0 \quad 0 \quad \alpha_1\ ]$$

$\cos(\omega t) = \cos(-\omega t)$

$\sin(\omega t) = -\sin(-\omega t)$

$U_k = e^{-j\frac{2\pi k}{N} n}$

$Re\{U_k\} = \cos$

$Im\{U_k\} = -\sin$

$\alpha_2 \sin(2\pi t f_2)$

$=$

$\alpha_2 e^{-j\phi} \cdot \cos(f_2 2\pi t)$

$\alpha_2 e^{-\frac{\pi}{2}} = j\alpha_2$

High freq vs Aliasing:

I can create a signal x'(t) which has the same x[n] and Fx[n]

aliasing

$$x'(t) = \alpha_1 \cos(2\pi \cdot 90\,Hz \cdot t) - \alpha_2 \sin(2\pi \cdot 60\,Hz \cdot t)$$

$$x''(t) = \alpha_1 \cos(2\pi\ 110\,Hz \cdot t) + \alpha_2 \sin(2\pi \cdot 140\,Hz \cdot t)$$

unrepresentable freq

(also technically aliasing, but without the complex conjugate)

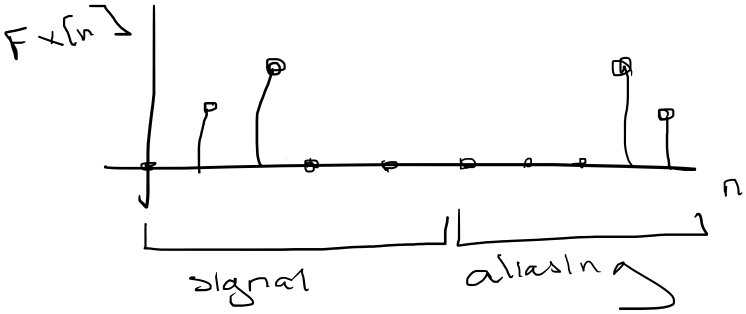
Ex2. Visual Aliasing

Helicopter floating!

II. Sampling Theorem

a. Nyquist Sampling Theorem

Let's agree that when we sample, all of our frequencies will be the positive frequency, and all of our aliasing is just a copy



Nyquist Sampling Theorem:

A bandlimited, cont time signal can be sampled and perfectly represented/ perfectly reconstructed from its samples, IF the sampling freq (f_s) is over twice as fast as the signals highest freq component (B)

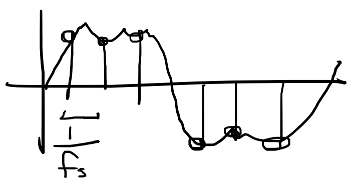
If you remember nothing else, remember this!

$$\boxed{f_s > 2B}$$

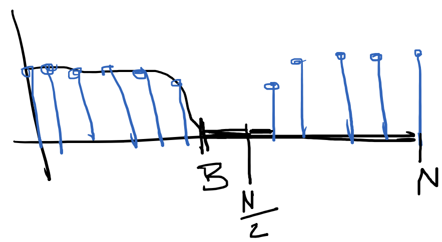
sampling freq — bandwidth

Note: strictly greater than

Bandwidth is the signal's highest frequency (all higher frequencies are zero)



$\Rightarrow$



Nyquist Frequency $\rightarrow \frac{f_s}{2}$

b. Sampling Freq and Sampling Window

Intuition

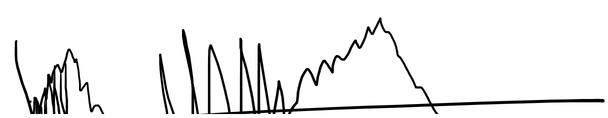
- Sample fast -> to get all the high, small freq details w/o aliasing

large f_s (small $T_s = \frac{1}{f_s}$)

- Collect data over a long time period, represent low frequencies

large N : $\frac{k}{N} = \frac{f}{f_s} \Rightarrow \frac{1}{N} = \frac{f_{min}}{f_s} \Rightarrow f_{min} = \frac{f_s}{N}$

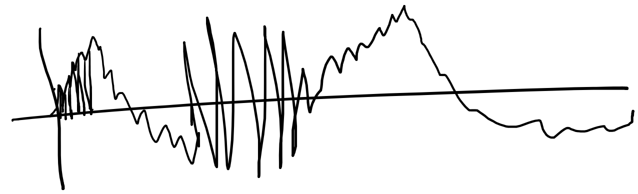
Best Life



Best life

super fast f_s

super long N



Number of samples: $N = f_s \text{ (samples/sec) } \frac{1}{f_{min}} \text{ (sec)}$

Ex Application Heart-Rate Sensor

Healthy avg human : 60-100 beats/min -> ~1beat/sec
Athlete : 40 beat/min -> 2/3 beat/sec
Fastest recorded : 480 beats/min -> 8 beats/sec

How fast must our ADC sample?

$$f_s > 2B$$

$$f_s > 2(8) > 16\,Hz$$

How many samples are needed

$$\frac{k}{N} = \frac{f_{min}}{f_s} \Rightarrow \frac{1}{N} = \frac{2/3}{16} \Rightarrow N \geq 24 \text{ samp}$$