Lecture 2

DFT of a square wave

2.1 Properties of the DFT

In this section, let x and y be time domain signals of length N and $X = Fx$ and $Y = Fx$ be their DFTs.

- The DFT is linear: if a and b are scalars, $F(ax + by) = aFx + bFy$.
- The DFT preserves energy: $\|Fx\|^2 = \|x\|^2$. [1]
- The DFT is conjugate-symmetric for real signals: if x is real, then $X[n] = \overline{X[-n]} = \overline{X[N-n]}$.

2.2 DFT of a rectangular pulse

Let $x \in \mathbb{C}^N$ be the following rectangular pulse, which approximates a square wave when $M = N/4$:

$$x[n] = \begin{cases} 1, & -M \leq n \leq M \\ 0, & \text{else} \end{cases} \tag{2.1}$$

Then the DFT of x is given by the following analysis equation:

$$X = Fx \tag{2.2}$$

This can be expanded in terms of the columns of F:

$$= \sum_{k=-M}^{M} \overline{u_k} \tag{2.3}$$

$$X[n] = \sum_{k=-M}^{M} \overline{u_k[n]} \tag{2.4}$$

$$= \frac{1}{\sqrt{N}} \sum_{k=-M}^{M} \omega^{-kn} \tag{2.5}$$

$$= \frac{1}{\sqrt{N}} \sum_{k=-M}^{M} \left(\omega^{n}\right)^{k} \tag{2.6}$$

[1] This is called Parseval's Theorem.

Use a formula for a finite geometric sum where the ratio is ω^n.

$$= \left(\frac{1}{\sqrt{N}}\right) \frac{(\omega^n)^{-M} - (\omega^n)^{M+1}}{1 - \omega^n} \tag{2.7}$$

Write each term in the big fraction as a power of ω_n.

$$= \left(\frac{1}{\sqrt{N}}\right) \frac{(\omega^n)^{-M} - (\omega^n)^{M+1}}{(\omega^n)^0 - (\omega^n)^1} \tag{2.8}$$

Cancel a factor of $(\omega_n)^{1/2}$ from the numerator and denominator of the big fraction. (This amounts to subtracting $1/2$ from each exponent.)

$$= \left(\frac{1}{\sqrt{N}}\right) \frac{(\omega^n)^{-M-1/2} - (\omega^n)^{M+1/2}}{(\omega^n)^{-1/2} - (\omega^n)^{1/2}} \tag{2.9}$$

Substitute the definition of ω_N.

$$= \left(\frac{1}{\sqrt{N}}\right) \frac{e^{-j\frac{2\pi}{N}n(M+1/2)} - e^{j\frac{2\pi}{N}n(M+1/2)}}{e^{-j\frac{2\pi}{N}n/2} - e^{j\frac{2\pi}{N}n/2}} \tag{2.10}$$

$$= \left(\frac{1}{\sqrt{N}}\right) \frac{e^{-j\frac{\pi}{N}(2M+1)n} - e^{j\frac{\pi}{N}(2M+1)n}}{e^{-j\frac{\pi}{N}n} - e^{j\frac{\pi}{N}n}} \tag{2.11}$$

$$= \left(\frac{1}{\sqrt{N}}\right) \frac{-2j\sin\frac{\pi}{N}(2M+1)n}{-2j\sin\frac{\pi}{N}n} \tag{2.12}$$

$$= \left(\frac{1}{\sqrt{N}}\right) \frac{\sin\frac{\pi}{N}(2M+1)n}{\sin\frac{\pi}{N}n} \tag{2.13}$$

Up to this point, we have used circular indexing in which e.g. $-n$ is interchangeable with $N-n$. Our forthcoming manipulation of this quotient will ruin its N-periodicity, so we will preemptively commit to 0-centered indexing $-\left\lceil\frac{N}{2}\right\rceil \ldots 0 \ldots \left\lceil\frac{N}{2}\right\rceil$.

$$= \left(\frac{1}{\sqrt{N}}\right) \frac{\sin\frac{\pi}{N}(2M+1)n}{\sin\frac{\pi}{N}n}, \qquad n = -\left\lceil\frac{N}{2}\right\rceil \ldots 0 \ldots \left\lceil\frac{N}{2}\right\rceil \tag{2.14}$$

The argument of $\sin$ in the denominator never gets very large. We can use the ignominous "small-angle approximation" $\sin x \approx x$.

$$\approx \left(\frac{1}{\sqrt{N}}\right) \frac{\sin\frac{2\pi}{N}(M+1/2)n}{\frac{\pi}{N}n}, \qquad n = -\left\lceil\frac{N}{2}\right\rceil \ldots 0 \ldots \left\lceil\frac{N}{2}\right\rceil \tag{2.15}$$

$$\approx \left(\frac{2M+1}{\sqrt{N}}\right) \frac{\sin\frac{\pi}{N}(2M+1)n}{\frac{\pi}{N}(2M+1)\,n}, \qquad n = -\left\lceil\frac{N}{2}\right\rceil \ldots 0 \ldots \left\lceil\frac{N}{2}\right\rceil \tag{2.16}$$

$$\approx \left(\frac{2M+1}{\sqrt{N}}\right) \operatorname{sinc}\left(\frac{2M+1}{N}n\right), \qquad n = -\left\lceil\frac{N}{2}\right\rceil \ldots 0 \ldots \left\lceil\frac{N}{2}\right\rceil \tag{2.17}$$

where $\operatorname{sinc} x = \lim_{t \to x} \frac{\sin \pi t}{\pi t}$ is an even function that evaluates to 1 at 0 and 0 at all other integers. Finally, if x is a square wave, then we can substitute $M = N/4$, resulting in the following.

$$X[n] \approx \left(\frac{2N/4+1}{\sqrt{N}}\right) \operatorname{sinc}\left(\frac{2N/4+1}{N} n\right), \qquad n = -\left\lceil \frac{N}{2} \right\rceil \dots 0 \dots \left\lceil \frac{N}{2} \right\rceil \tag{2.18}$$

$$\approx \frac{\sqrt{N}}{2} \operatorname{sinc}\left(\frac{1}{2} n\right), \qquad n = -\left\lceil \frac{N}{2} \right\rceil \dots 0 \dots \left\lceil \frac{N}{2} \right\rceil \tag{2.19}$$

Therefore the DFT of a square wave (half on, half off) is a sinc function that crosses zero every other frequency.

"The DFT of a square is a sinc and the DFT of a sinc is a square."

If x is a square wave and X is its matching sinc, then the following analysis equation holds:

$$X = Fx \tag{2.20}$$

Conjugating both sides and using the fact that both x and X are real,

$$\overline{X} = \overline{Fx} \tag{2.21}$$

$$= \overline{F}\overline{x} \tag{2.22}$$

$$X = \overline{F}x \tag{2.23}$$

Using the fact that F is both unitary and symmetric, we can substitute $\overline{F} = F^*$.

$$X = F^* x \tag{2.24}$$

A sinc is the DFT of a square wave, but it is the inverse DFT of a square wave as well! Multiplying through by F yields an analysis equation.

$$FX = x \tag{2.25}$$

This kind of pairing relationship exists whenever x and X are both real. Another example is that the DFT of a DC signal is an impulse (one 1 and $N-1$ zeros), and that the DFT of an impulse is DC.