Tracking Control

7/30

Thursday, July 30

- CCF Continued
- Tracking Control
- Module Recap

CCF:

$$A_c := \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix} \qquad B_c := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

If original system is controllable,

$\exists$ T which brings $z = Tx$ to system in CCF.

i.e.
$$TAT^{-1} = A_c$$
$$TB = B_c$$

Then, $u = -K_c z$

$u = -Kx$, $K = K_c T$

$$T(A-BK)T^{-1} = A_c - B_c K_c$$

Note that

$$A-BK \quad \text{and} \quad A_c - B_c K_c$$

have same eigenvalues.

$$C := \begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix}$$

C^{-1} exists since system is controllable

Let $q^T =$ first row of C^{-1}.

$C^{-1}C = I$

$$q^T C = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$
$$= \begin{bmatrix} q^T A^{n-1} B & q^T A^{n-2} B & \cdots & q^T B \end{bmatrix}$$

$$\phantom{q^T A^{n-1} B \quad} 0 \quad 0 \quad 0$$

Idea follows:

$$T := \begin{bmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{bmatrix}$$

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$$TB = \begin{bmatrix} q^T B \\ q^T A B \\ \vdots \\ q^T A^{n-1} B \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{B_c} =$$

$TAT^{-1} = A_c$?

$$TA = A_c T$$

$$TA = \begin{bmatrix} q^T A \\ q^T A^2 \\ \vdots \\ q^T A^n \end{bmatrix} \qquad A_c T = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ a_1 & \cdots & \cdots & \cdots & a_n \end{bmatrix} \begin{bmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} q^T A \\ q^T A^2 \\ \vdots \\ q^T \left(\underline{a_1 I + a_2 A + \cdots + a_n A^{n-1}} \right) \end{bmatrix}$$

Aside:

For a matrix A:

If characteristic polynomial

is $x^n - a_n x^{n-1} - \dots - a_1$

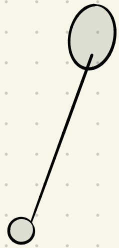
Then:

$$\Rightarrow \quad A^n - a_n A^{n-1} \dots - a_1 I = 0$$

"Cayley - Hamilton Theorem"

$$A^n = a_n A^{n-1} + \dots + a_1 I$$

Exercise: Stabilize a Pendulum



$$\hat{X}(t) = \begin{bmatrix} X_2(t) \\ -\frac{K}{m} X_2(t) - \frac{g}{l}\sin X_1(t) + \frac{u(t)}{ml} \end{bmatrix}$$

Linearize around upright eq. point

$u^* = 0 \qquad x^* = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

$\delta x(t) := x(t) - x^*$

$\delta u(t) := u(t) - u^* = u(t)$

$$\delta \dot{x}(t) \simeq \begin{bmatrix} 0 & 1 \\ g/l & -K/m \end{bmatrix} \delta x(t) + \begin{bmatrix} 0 \\ \frac{1}{\cancel{ml}} \end{bmatrix} \delta u(t)$$

$$\lambda = \frac{-K}{2m} \pm \underbrace{\frac{1}{2}\sqrt{\left(\frac{K}{m}\right)^2 + 4\left(\frac{g}{l}\right)}}$$

$$\widehat{\delta u}(t) := \frac{1}{ml}\,\delta u(t)$$

$$\delta \dot{x}(t) = \underbrace{\begin{array}{cc} \begin{bmatrix} 0 & 1 \\ g/l & -K/m \end{bmatrix} \\ \uparrow \quad\quad \uparrow \\ a_1 \quad\quad a_2 \end{array}}_{A} \delta x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \widehat{\delta u}(t)$$

Characteristic Polynomial of

$(A-BK)$

$\hookrightarrow \lambda^2 - (a_2 - K_2)\lambda - (a_1 - K_1) = 0$

$$\lambda^2 - \left(-\frac{k}{m} - K_2\right)\lambda - \left(g/\ell - K_1\right) = 0$$

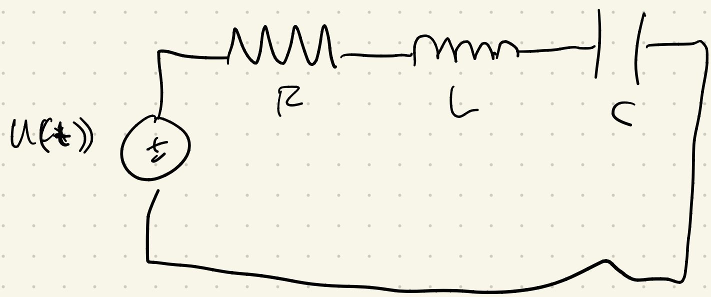
$$(\lambda+1)(\lambda+0.5) = \lambda^2 + 1.5\lambda + 0.5$$

$$K_2 + \frac{k}{m} = 1.5$$

$$K_1 - g/\ell = 0.5$$

Tracking Control

Example: RLC



$x_1 = V_c$

$x_2 = i_l$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u(t)$$

Let $R = 2.5$

$L = 1$

$C = 1$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2.5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda\left(\begin{bmatrix} 0 & 1 \\ -1 & -2.5 \end{bmatrix}\right)$$

$$\lambda^2 + 2.5\lambda + 1 = 0$$

$$\lambda_1 = -0.5$$

$$\lambda_2 = -2$$

$$V_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$V = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \qquad V^{-1} = \begin{bmatrix} -2/3 & -1/3 \\ -1/3 & -2/3 \end{bmatrix}$$

$$X_{k+1} = V \underbrace{\begin{bmatrix} e^{-T/2} & \\ & e^{-2T} \end{bmatrix} V^{-1}}_{Ad} X_k + \underbrace{V \begin{bmatrix} \frac{2}{3}(e^{-T/2}-1) \\ \frac{1}{3}(e^{-2T}-1) \end{bmatrix}}_{Bd} u_k$$

$$T = 0.2$$

$$Ad = \begin{bmatrix} 0.983 & 0.1563 \\ 0.1563 & 0.5921 \end{bmatrix} \quad Bd = \begin{bmatrix} 0.017 \\ 0.1563 \end{bmatrix}$$

$$x_{k+1} = A_d x_k + B_d u_k$$

First assume no control:

Start with $x_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$x_k = A_d^k x_0$$

→ See Matlab

$$\min_{u_0, x_1, u_1, x_2, u_2, x_3} \sum_{t=1}^{3} \|x_t - \hat{x}_t\|_2^2 + \sum_{t=0}^{2} \|u_0\|_2^2$$

such that
$$\left.\begin{aligned} x_1 &= Ax_0 + Bu_0 \\ x_2 &= Ax_1 + Bu_1 \\ x_3 &= Ax_2 + Bu_2 \end{aligned}\right\}$$

$\hat{x}_k$ = reference trajectory

$$\min_{u_0, x_1, u_1, \dots, x_3} \sum \|u_t\|_2^2 + \sum \|x_t - \hat{x}_t\|_2^2$$

s.t.

$$\begin{bmatrix} B & -I & & & & \\ & A & B & -I & & \\ & & & A & B & -I \end{bmatrix} \begin{bmatrix} u_0 \\ x_1 \\ u_1 \\ x_2 \\ u_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -Ax_0 \\ 0 \\ 0 \end{bmatrix}$$

Let $d_k := x_k - \hat{x}_k$

$x_k := d_k + \hat{x}_k$

$$\boxed{\min_{u_0, d_1, u_1, d_2, u_2, d_3} \sum_{t=0}^{2} \|u\|_2^2 + \sum_{t=1}^{3} \|d_t\|_2^2 \qquad \begin{bmatrix} B & -I & & & & \\ & A & B & -I & & \\ & & & A & B & -I \end{bmatrix} \begin{bmatrix} u_0 \\ d_1 \\ u_1 \\ d_2 \\ u_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} \hat{x}_1 - Ax_0 \\ \hat{x}_2 - A\hat{x}_1 \\ \hat{x}_3 - A\hat{x}_2 \end{bmatrix}}$$

$$\min_{\begin{bmatrix} u_0 \\ d_1 \\ u_1 \\ d_2 \\ u_2 \\ d_3 \end{bmatrix}} \left\| \begin{bmatrix} u_0 \\ d_1 \\ u_1 \\ d_2 \\ u_2 \\ d_3 \end{bmatrix} \right\|^2$$

$$\text{s.t.} \quad \underbrace{\begin{bmatrix} B & -I & & & & \\ & A & B & -I & & \\ & & & A & B & I \end{bmatrix}}_{\text{Full row-rank}} \begin{bmatrix} u_0 \\ d_1 \\ u_1 \\ d_2 \\ u_2 \\ d_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{x}_1 - A x_0 \\ \hat{x}_2 - A \hat{x}_1 \\ \hat{x}_3 - A \hat{x}_2 \end{bmatrix}}$$

Can solve for

$$\begin{bmatrix} u_0^* \\ d_0^* \\ \vdots \\ d_3^* \end{bmatrix}$$

using least-squares solution.

$$x_k^* = d_k^* + \hat{x}_k$$

Note that the problem on previous page is in "Standard" Least-Squares form

$$\min_{x} \|x\|_2^2 \quad \text{s.t.} \quad Ax = y$$

We could choose to weight controls and reference error differently by using C Matrix as in homework:

$$\min_{x} \|Cx\|_2^2 \quad \text{s.t.} \quad Ax = y$$

Now (x_k^*, u_k^*) found form a dynamically feasible trajectory:

$$x_{k+1}^* = A x_k^* + B u_k^*$$

$$x_{k+1} = A x_k + B u_k$$

If we apply u^*, our system should follow x^*. What if we have some modeling error though?

$$(x_{k+1} - x_{k+1}^*) = A(x_k - x_k^*) + B(u_k - u_k^*)$$

If we choose $(u_k - u_k^*) \equiv 0$

Then If A is stable

Then $x_k \to x_k^*$

If not, we can

Define $(u_k - u_k^*) = -K(x_k - x_k^*)$

so that closed-loop system is stable with respect to time-varying trajectory (x_k^*)

$$u_k = \underbrace{u_k^* + K x_k^*}_{\text{Open-loop control}} - \underbrace{K x_k}_{\text{feedback / closed-loop control}}$$