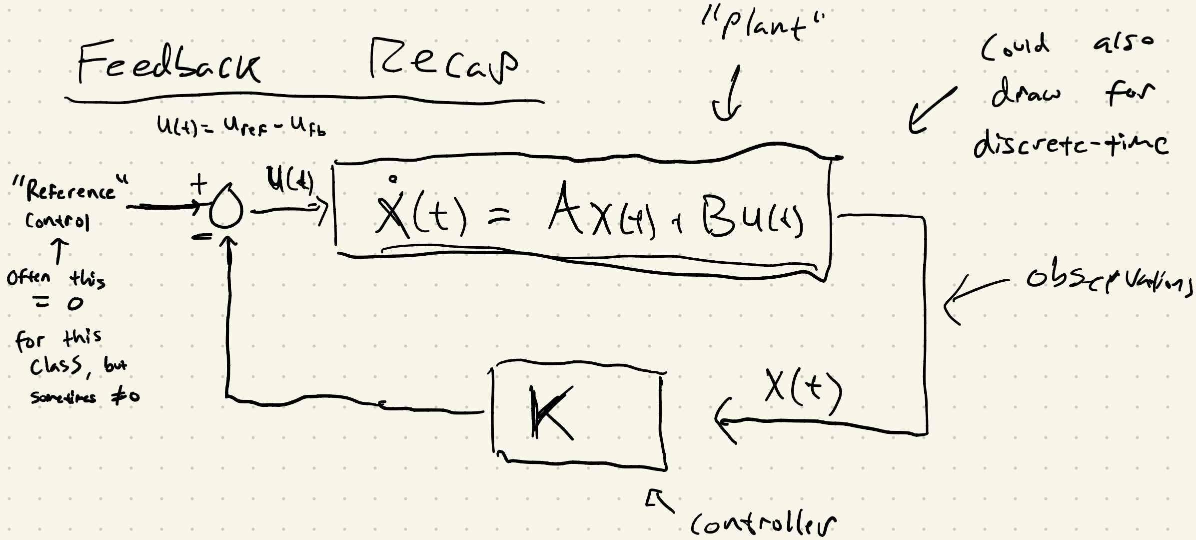
Controllable Canonical Form

July 29

Wednesday, July 29th

- Feedback Control Recap
- Controllable Canonical form

Feedback Recap



Continuous-time:

$x \in \mathbb{R}^n$
$u \in \mathbb{R}^m$

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$\Downarrow \quad u(t) = -Kx(t)$$

$$\dot{x}(t) = (A - BK)x(t) \qquad K \in \mathbb{R}^{m \times n}$$

Discrete-time:

$$x_{k+1} = Ax_k + Bu_k$$

$$\Downarrow$$

$$x_{k+1} = (A - BK)x_k$$

Example: Cruise Control

$$\dot{v}(t) = \frac{-1}{2M} \rho a c \, v(t)^2 + \frac{1}{RM} u(t)$$

$$u^*(v^*) = \frac{R}{2} \rho a c (v^*)^2$$

$$\delta v(t) = v(t) - v^*$$

$$\delta u(t) = u(t) - u^*$$

$$\delta \dot{v}(t) \simeq \lambda \, \delta v(t) + b \, \delta u(t)$$

$$\lambda = \frac{-1}{M} \rho a c v^* \qquad b = \frac{1}{RM}$$

$\lambda < 0 \Rightarrow$ System is Stable

If we choose $\delta u(t) \equiv 0 \quad \forall t$

$(u(t) = u^*)$

$$\delta \dot{v}(t) = \lambda \, \delta v(t) \qquad \delta v(t) \to 0$$

$$v(t) \to v^*$$

$\lambda = -0.01 \, s^{-1}$

How about:

$$\delta u(t) = -K\,\delta v(t)$$

$$\hookrightarrow \quad \delta \dot{v}(t) = \underbrace{(\lambda - bK)}_{= -1 \text{ instead of } -0.01}\,\delta v(t)$$

$$\delta u(t) = -K\,\delta v(t)$$

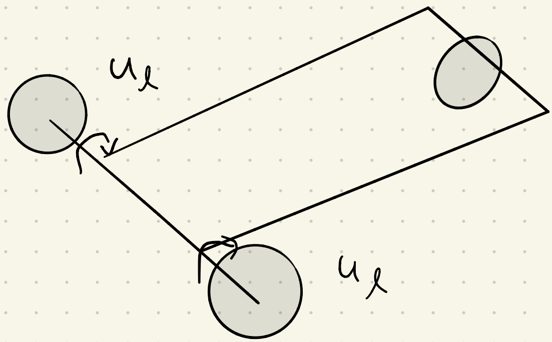
$$u(t) = \underset{\uparrow}{u^*} - K(v(t) - v^*)$$

reference control

Feedback control

$$= \underbrace{(u^* + K v^*)}_{\text{reference control}} - \underbrace{K v(t)}_{\text{feedback control}}$$

Lab Car:



$$d_\ell[k+1] = d_\ell[k] + \theta_\ell u_\ell[k] - \beta_\ell$$

$$d_r[k+1] = d_r[k] + \theta_r u_r[k] - \beta_r$$

$u_\ell[k] = u_r[k] \Rightarrow$ bad idea due to different parameters

Propose:

$$u_\ell[k] = \frac{V^* + \beta_\ell}{\theta_\ell} + \frac{k_\ell}{\theta_\ell}\left(d_\ell[k] - d_r[k]\right)$$

$$u_r[k] = \frac{V^* + \beta_r}{\theta_r} + \frac{k_r}{\theta_r}\left(d_\ell[k] - d_r[k]\right)$$

V^* desired wheel velocity

$$\begin{bmatrix} d_\ell[k+1] \\ d_r[k+1] \end{bmatrix} = \begin{bmatrix} d_\ell[k] \\ d_r[k] \end{bmatrix} + \begin{bmatrix} v^* \\ v^* \end{bmatrix} + \begin{bmatrix} K_\ell(d_\ell[k]-d_r[k]) \\ K_r(d_\ell[k]-d_r[k]) \end{bmatrix}$$

$$\Delta[k] := (d_\ell[k] - d_r[k])$$

$$\Delta[k+1] = \Delta[k] + (K_\ell - K_r)\Delta[k]$$

$$\Delta[k+1] = (1 + K_\ell - K_r)\Delta[k]$$

$$\Delta[k] \to 0$$

as $\text{time} \to \infty$

choose K_ℓ, K_r such that

$$|1 + K_\ell - K_r| < 1$$

But if $K_\ell = K_r = 0$

$$\Delta[k+1] = \Delta[k] \quad \forall k$$

Controllable Canonical form

$$X_{t+1} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} X_t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t$$

$$\lambda^2 - a_2\lambda - a_1 = 0$$

$$X_{t+1} = \begin{bmatrix} 0 & 1 \\ a_1 - k_1 & a_2 - k_2 \end{bmatrix} X_t$$

$$\lambda^2 - (a_2 - k_2)\lambda - (a_1 - k_1) = 0$$

$$X_{t+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 & a_2 & a_3 \end{bmatrix} X_t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_t$$

$$\lambda^3 - a_3\lambda^2 - a_2\lambda - a_1 = 0$$

FB: $A - Bu = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 - k_1 & a_2 - k_2 & a_3 - k_3 \end{bmatrix}$

$$\lambda^3 - (a_3 - k_3)\lambda^2 - (a_2 - k_2)\lambda - (a_1 - k_1)$$

$$A_c := \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \end{bmatrix}$$

$$B_c := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Characteristic polynomial of A_c

$$\lambda^n - a_n \lambda^{n-1} - a_{n-1} \lambda^{n-2} - \cdots - a_1$$

$(A_c - B_c K_c)$ has characteristic polynomial

$$\lambda^n - (a_n - K_n)\lambda^{n-1} - (a_{n-1} - K_{n-1})\lambda^{n-2} - \cdots - (a_1 - K_1)$$

Claim: If some matrices A, B define a controllable system, then $\exists$ a transformation T which brings A to A_c
B to B_c

i.e. $TAT^{-1} = A_c$

$TB = B_c$

$$\boxed{\begin{aligned} &T(A-BK)T^{-1} = \\ &TAT^{-1} - TBKT^{-1} \\ &= A_c - B_c K_c \end{aligned}}$$

If $z \to 0$
$x \to 0$

Define $z = Tx \Rightarrow x = T^{-1} z$

$$\dot{z}(t) = A_c z(t) + B_c u(t)$$

or

$$z_{t+1} = A_c z_t + B_c u_t$$

$$u = -K_c z = -K_c T x$$

$= -Kx$ where $\underline{K = K_c T}$

$KT^{-1} = K_c$

First: Note that

A has eigenvalues same

as TAT^{-1} for invertible T

$\underline{Av = \lambda v}$

$TAT^{-1} \cdot Tv = T\underline{Av} = \lambda Tv$

$\Rightarrow$ λ is eigenvalue of both A and TAT^{-1} where v is corresponding evec for A and Tv is corresponding evec for (TAT^{-1}).

$$\overbrace{T(A-BK)T^{-1}}^{(A_c - B_c K_c)} \quad \text{and} \quad (A-BK)$$

have same eigenvalues and same characteristic polynomial.

How do we know T exists?

A, B are Controllable

$$\Rightarrow \quad C := \left[A^{n-1}B \;\; A^{n-2}B \; \cdots \; B \right]$$

is full-rank and invertible

That means inverse C^{-1} exists.

Let q^T be first row of C^{-1}

$$\Rightarrow \quad q^T C = \left[1 \;\; 0 \; \cdots \; 0 \right]$$

$$T = \begin{bmatrix} q^T \\ q^T A \\ q^T A^2 \\ \vdots \\ q^T A^{n-1} \end{bmatrix}$$