Feedback

July 28

Tuesday, July 28th

- Recap Stability
- Predicting transients from eigenvalues
- Feedback/Stabilizing Control!

Stability:

Discrete-time Linear System

$x \in \mathbb{R}^n$
$u \in \mathbb{R}^m$

$$x_{k+1} = A x_k + B u_k + c$$

Stability $\Longleftrightarrow$ $|\lambda_i| < 1 \quad \forall\, i \in \{1, \dots, n\}$

λ_i eigenvalues of A

Instability $\Longleftrightarrow$ $\exists\, i \in \{1, \dots, n\} : |\lambda_i| > 1$

Continuous-time Linear System

$x \in \mathbb{R}^n$
$u \in \mathbb{R}^m$

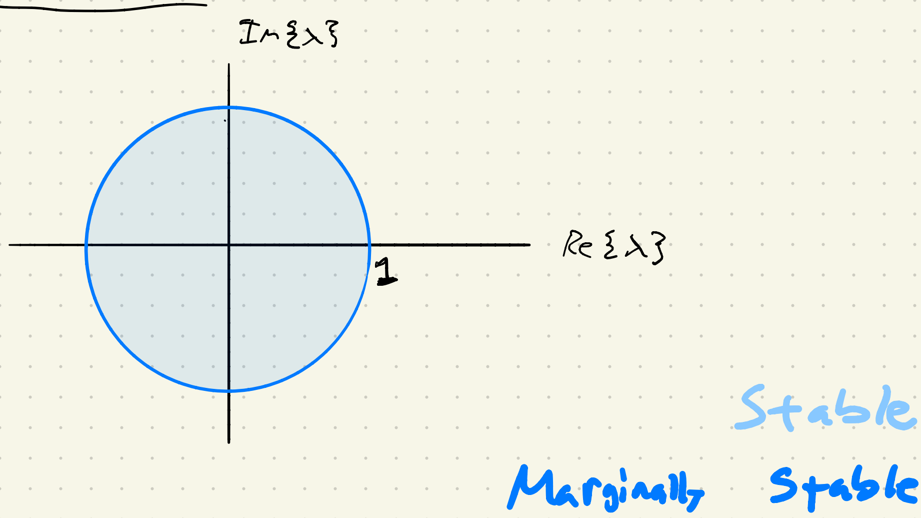
$$\dot{x}(t) = A x(t) + B u(t) + c$$

Stability $\Longleftrightarrow$ $\mathrm{Re}\{\lambda_i\} < 0 \quad \forall\, i \in \{1, \dots, n\}$

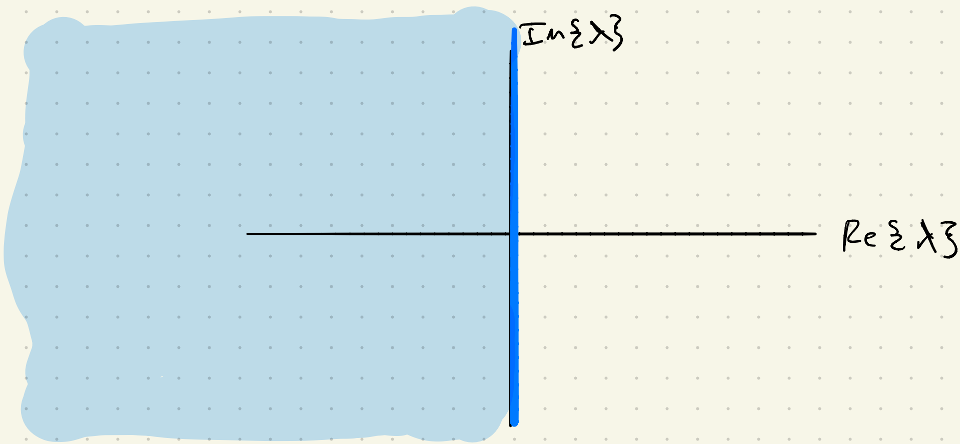
λ_i eigenvalues of A

Instability $\Longleftrightarrow$ $\exists\, i \in \{1, \dots, n\} : \mathrm{Re}\{\lambda_i\} > 0$

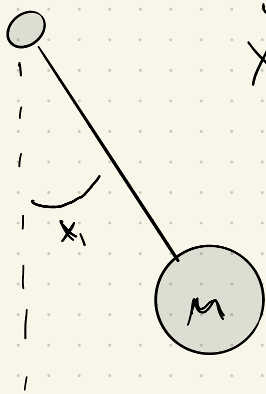
Discrete-time:



Continuous-time:



Example: Pendulum (no control)



$$\dot{x}(t) = \begin{bmatrix} x_2(t) \\ -\frac{k}{m} x_2(t) - \frac{g}{\ell} \sin x_1(t) \end{bmatrix}$$

Linearized System:

$$\delta \dot{x}(t) \simeq \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} \cos x_1^* & -\frac{k}{m} \end{bmatrix} \delta x(t)$$

$$x^*_{up} = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \qquad x^*_{down} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

UP:

$$\delta \dot{x}(t) \simeq \begin{bmatrix} 0 & 1 \\ g/\ell & -k/m \end{bmatrix} \delta x(t)$$

$$\lambda = \frac{-\frac{k}{m} \pm \sqrt{\left(\frac{k}{m}\right)^2 + 4\left(\frac{g}{\ell}\right)}}{2}$$

$$\lambda\left(\lambda + \frac{k}{m}\right) - g/\ell = 0$$

$$\lambda^2 + \frac{k}{m}\lambda - g/\ell = 0$$

Down:

$$\delta \dot{x}(t) \simeq \begin{bmatrix} 0 & 1 \\ -g/\ell & -k/m \end{bmatrix} \delta x(t)$$

$$\lambda^2 + \frac{k}{m}\lambda + \frac{g}{\ell} = 0$$

$$\lambda = \frac{-\frac{k}{m} \pm \sqrt{\left(\frac{k}{m}\right)^2 - 4\frac{g}{\ell}}}{2}$$

Transients and Eigenvalues

$$AV = V\Lambda$$

$$x_{k+1} = A x_k + B u_k$$

define $\quad z_k = V^{-1} x_k$

$$x_k = V z_k$$

$$z_{k+1} = V^{-1} A V z_k + V^{-1} B u_k$$

$$= \Lambda z_k + V^{-1} B u_k$$

$$z_i[k+1] = \lambda_i z_i[k] + b_i u[k]$$

$$\underbrace{z_i[k] = \lambda_i^k z_i[0]} + \sum_{t=0}^{k-1} \lambda_i^{k-1-t} b_i u[t]$$

Write $\lambda_i = |\lambda_i| e^{j\omega}$ (polar form)

$$\lambda_i^t = |\lambda_i|^t e^{j\omega t} = \underbrace{|\lambda_i|^t \cos(\omega t)}_{\text{Real}} + j \underbrace{|\lambda_i|^t \sin(\omega t)}_{\text{imaginary}}$$

$$x[k] = V z[k]$$

Continuous-time (diagonalizable)

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$z(t) = V^{-1}x(t)$$

$$\dot{z}(t) = \Lambda\, z(t) + V^{-1}B\,u(t)$$

$$\underbrace{z_i(t) = e^{\lambda t} z_i(0)} + \begin{pmatrix}\text{influence} \\ \text{of control}\end{pmatrix}$$

$$\lambda = \alpha + j\omega$$

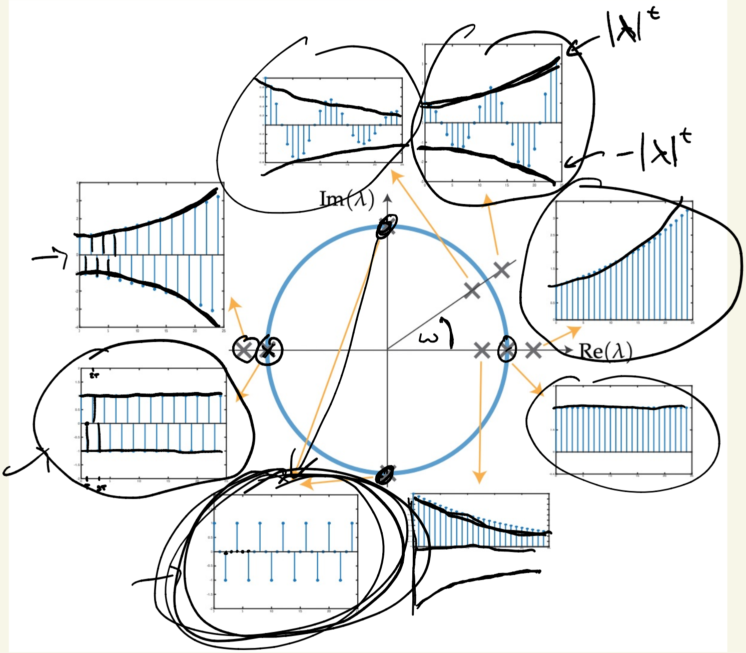
$$e^{\lambda t} = e^{\alpha t} e^{j\omega t} = \underbrace{e^{\alpha t}\cos(\omega t)}_{\text{real}} + j\underbrace{e^{\alpha t}\sin(\omega t)}_{\text{imaginary}}$$

$$z_i(t) = z_i(0)\,e^{\alpha t}\cos(\omega t) + j\, z_i(0)\,e^{\alpha t}\sin(\omega t)$$

(If $z_i(0)$ is real)

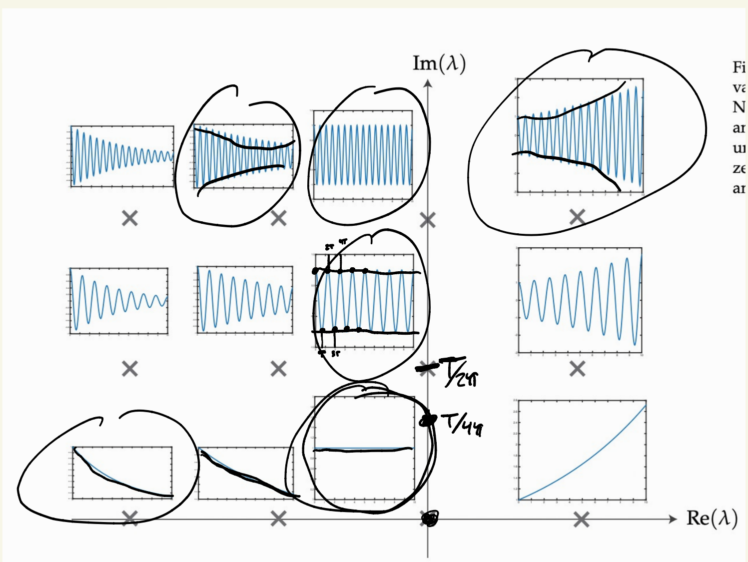
$$x(t) = V z(t)$$

Discrete-time

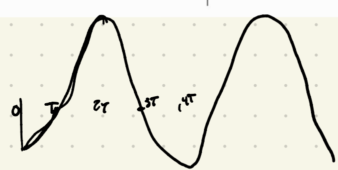


λ^t

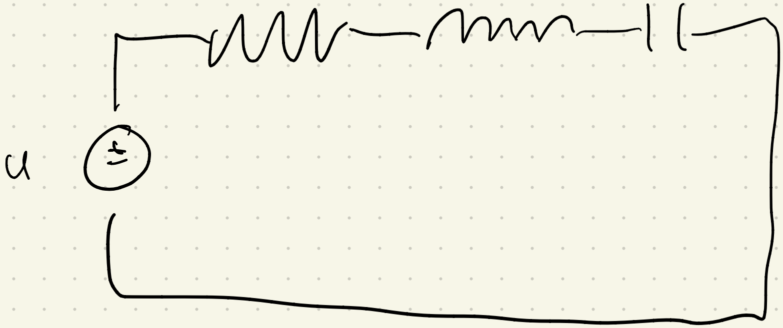
Continuous-time



$e^{\lambda t}$



Example: RLC Circuit



$$X(t) = \begin{bmatrix} V_c \\ i_\ell \end{bmatrix}$$

$R > 0$

$L > 0$

$C > 0$

$$\dot{X}(t) = \begin{bmatrix} 0 & 1/C \\ \frac{-1}{L} & -R/L \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u(t)$$

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

$$\alpha = \frac{R}{2L}$$

$$\lambda = \frac{-R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$\alpha > \omega_0 \Rightarrow$ eigenvalues real, negative

$\alpha < \omega_0 \Rightarrow \lambda = -\alpha \pm j\omega$

$$\omega = \sqrt{\omega_0^2 - \alpha^2}$$

Feedback!

$x_k \in \mathbb{R}^n$
$u_k \in \mathbb{R}^m$

$$x_{k+1} = A x_k + B u_k$$

Consider: we want to reach

$x = 0 \in \mathbb{R}^n$ from any

initial config

Define a control policy:

$$\underline{u_k = -K x_k} \qquad K \in \mathbb{R}^{m \times n}$$

$$x_{k+1} = (A - BK) x_k$$

We can choose values of K

such that eigenvalues of

$(A-BK)$ have magnitude less

than $\underline{1}$

If the system is controllable, we can choose K to achieve <u>any</u> eigenvalues we want.

<u>Ex:</u>

$$X_{k+1} = \underbrace{\begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}}_{A} x_k + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u_k$$

$$\det(\lambda I - A) = \lambda^2 - a_2\lambda - a_1 = 0$$

$$u_k = -K x_k \qquad K = [k_1 \;\; k_2]$$

$$\Rightarrow X_{k+1} = \underbrace{\begin{bmatrix} 0 & 1 \\ a_1 - k_1 & a_2 - k_2 \end{bmatrix}}_{(A-BK)} x_k$$

$$\lambda^2 - \underbrace{(a_2 - k_2)}\lambda - \underbrace{(a_1 - k_1)} = 0$$

we desire eigenvalues $\hat{\lambda}_1, \hat{\lambda}_2 \Rightarrow (\lambda - \hat{\lambda}_1)(\lambda - \hat{\lambda}_2) = 0$

$$\lambda^2 - \underbrace{(\hat{\lambda}_1 + \hat{\lambda}_2)}\lambda + \underbrace{\hat{\lambda}_1\hat{\lambda}_2} = 0$$

$$\hat{\lambda}_1 + \hat{\lambda}_2 = a_2 - k_2$$

$$-\hat{\lambda}_1\hat{\lambda}_2 = a_1 - k_1$$

$$\Rightarrow \quad k_1 = a_1 + \hat{\lambda}_1\hat{\lambda}_2$$

$$k_2 = a_2 - \hat{\lambda}_1 - \hat{\lambda}_2$$

Ex 2:

$$X[k+1] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} X[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[k]$$

$$(A - BK) = \begin{bmatrix} 1-k_1 & 1-k_2 \\ 0 & 2 \end{bmatrix}$$

$$\lambda_1 = 1 - k_1$$

$$\lambda_2 = 2$$

$$X_2[k+1] = 2X_2[k]$$

Continuous-time:

$x \in \mathbb{R}^n$
$u \in \mathbb{R}^m$

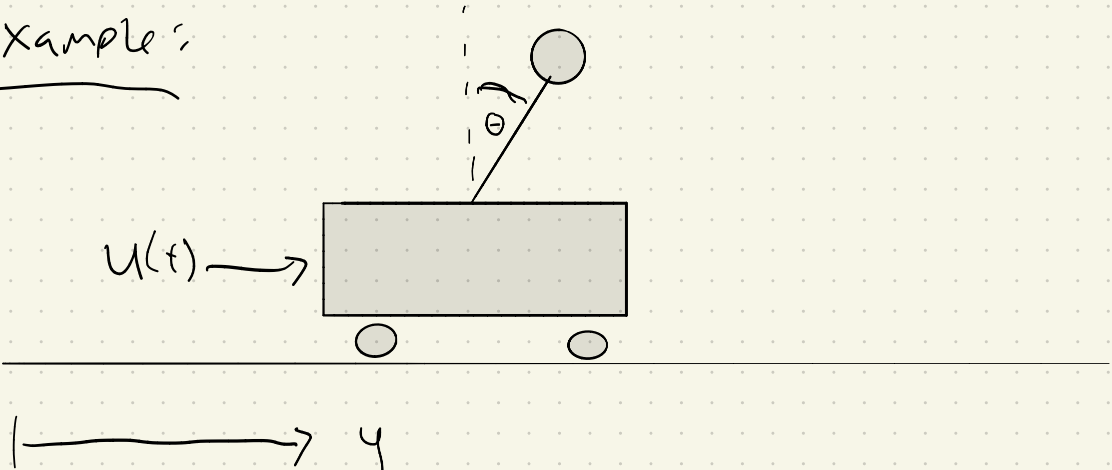
$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

define $\quad u(t) := -K\,x(t)$

$$\dot{x}(t) = (A - BK)\,x(t)$$

Choose K such that eigenvalues of $(A - BK)$ have negative real part

Example:



$$X(t) := \begin{bmatrix} \theta \\ \dot{\theta} \\ \dot{y} \end{bmatrix}$$

$$\dot{X}(t) = \begin{bmatrix} X_2(t) \\ f_2(X(t), u(t)) \\ f_3(X(t), u(t)) \end{bmatrix}$$

Linearize around $X^* = [0, 0, 0]^T$

$$\delta\dot{X}(t) \simeq \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ \frac{M+m}{Ml} g & 0 & 0 \\ \frac{-m}{M} g & 0 & 0 \end{bmatrix}}_{A} \delta X(t) + \underbrace{\begin{bmatrix} 0 \\ \frac{-1}{M} l \\ \frac{1}{M} \end{bmatrix}}_{B} \delta u(t)$$

choose $M=1$ $\quad l=1$

$m=0.1$ $\quad g=10$

$$\delta u(t) := -K\,\delta x(t)$$

$$A-BK = \begin{bmatrix} 0 & 1 & 0 \\ 11 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} K$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 11+K_1 & K_2 & K_3 \\ -1-K_1 & -K_2 & -K_3 \end{bmatrix}$$

$$\lambda^3 + (K_3 - K_2)\lambda^2 + (-K_1 - 11)\lambda - 10K_3 = 0$$

$$(\lambda - \hat{\lambda}_1)(\lambda - \hat{\lambda}_2)(\lambda - \hat{\lambda}_3) = 0$$

$\Rightarrow$ Match coefficients

choose K_1, K_2, K_3 accordingly