Stability

7/27

Monday, July 27^{th}

Today:
- Stability
  - Scalar discrete-time
  - Vector discrete-time
  - Continuous-time
- Maybe: Upper-Triangularization

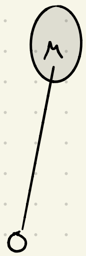
Intro:

This week we will be talking about stability and stabilizing control.

Often times useful operating points of systems are unstable, and we must design controls to stabilize the system.

E.g.

Example:



Pendulum:

$\delta x(t) := x(t) - \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

$\delta u(t) := u(t) - 0$

Linearize around upright $\begin{bmatrix} \pi \\ 0 \end{bmatrix}$
equilibrium:

$$\delta \dot{x}(t) \approx \begin{bmatrix} 0 & 1 \\ \frac{g}{\ell} & -\frac{k}{M} \end{bmatrix} \delta x(t) + \begin{bmatrix} 0 \\ \frac{1}{M\ell} \end{bmatrix} \delta u(t)$$

Discretize:

$$\delta x_{k+1} = A_d \, \delta x_k + B_d \, \delta u_k \leftarrow$$

System is controllable.

$x_1(t) = \pi - 0.1$

$x_2(t) = 0$

Assume

$$\delta x_0 = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix} = x(t) - \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

$\Rightarrow$ Could solve optimal control problem to get to $\delta x_k \approx \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, but will that work?

In order to learn how to address this, we must first study Stability.

Stability:

$x_k \in \mathbb{R}, \quad u_k \in \mathbb{R}$

Consider $\underline{x_{k+1} = \lambda x_k + b u_k}$

$$x_k = \lambda^k x_0 + \lambda^{k-1} b u_0 + \lambda^{k-2} b u_1 + \dots + b u_{k-1}$$

$$= \lambda^k x_0 + \sum_{t=0}^{k-1} \lambda^{k-1-t} b u_t \quad \text{for } k = 1, 2, \dots,$$

Definition:

This system is Stable if the state remains bounded for any initial configuration and any bounded input sequence

Conversely: System is Unstable if we can find any initial config. or any bounded input sequence such that $|x_k| \to \infty$ as $k \to \infty$

For Scalar case:

If $|\lambda| > 1$:

choose $u_k \equiv 0 \quad \forall k$

$$|x_k| = |\lambda^k x_0| \rightarrow \infty$$

as $k \rightarrow \infty$

$\Rightarrow$ System unstable

If $|\lambda| < 1$:

$$|x_k| = \left|\lambda^k x_0 + \sum_{t=0}^{k-1} \lambda^{k-1-t} b u_t\right|$$

$$\leq |\lambda^k x_0| + \left|\sum_{t=0}^{k-1} \lambda^{k-1-t} b u_t\right|$$

$$\leq |\lambda^k x_0| + |b| \sum_{t=0}^{k-1} |\lambda^{k-1-t}| \, |u_t|$$

$$\leq |\lambda^k x_0| + |b| M \sum_{t=0}^{k-1} |\lambda^{k-1-t}|$$

$$\leq |x_0| + |b| M \sum_{t=0}^{k-1} |\lambda|^t$$

$$\leq |x_0| + |b| M \frac{1}{1-|\lambda|}$$

$\Rightarrow$ If $|\lambda| < 1$

$\Rightarrow$ System remains bounded $\Rightarrow$ Stable

What if λ is complex?

$$|\lambda| = \sqrt{Re\{\lambda\}^2 + Im\{\lambda\}^2}$$

What if $|\lambda| = 1$

If no control:

$$|x_k| = |\lambda^k x_0| = |x_0|$$

Eg. $\lambda = 1$, define $u_k \equiv 1 \quad \forall k$

$$\sum_{t=0}^{k-1} \lambda^{k-1-t} b u_t = k \cdot b \to \infty$$

as $k \to \infty$

If $|\lambda| = 1$:

we call this Marginally Stable

Recap:

$|\lambda| < 1 \Rightarrow$ Stable

$|\lambda| > 1 \Rightarrow$ Unstable

$|\lambda| = 1 \Rightarrow$ Marginally Stable

Vector case:

$x_k \in \mathbb{R}^n$

Consider $u_k \in \mathbb{R}^m$

$$x[k+1] = Ax[k] + Bu[k]$$

$$x[k] = A^k x[0] + \sum_{t=0}^{k-1} A^{k-1-t} B u[t] \quad k = 1, 2, \ldots$$

$$A^k := \underbrace{A \cdot A \cdot A}_{k \text{ times}} \cdots$$

First assume:

A diagonalizable

$$AV = V\Lambda \Rightarrow V^{-1}AV = \Lambda$$

$$z[k] = V^{-1}x[k]$$
$$x[k] = V\,z[k]$$

$$z[k+1] = \underbrace{\Lambda}_{A_{new}} z[k] + \underbrace{V^{-1}B}_{B_{new}} u[k]$$

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$z_i[k+1] = \lambda_i z_i[k] + b_i u[k]$$

b_i i^{th} row of B_{new}

For each scalar subsystem

$|\lambda_i| < 1 \Rightarrow$ stable

$|\lambda_i| > 1 \Rightarrow$ unstable

$|\lambda_i| = 1 \Rightarrow$ marginally stable

For whole system

If $\exists\, i \in \{1, \ldots, n\}$

with $|\lambda_i| > 1 \Rightarrow$ unstable

If $|\lambda_i| < 1 \;\; \forall i \Rightarrow$ stable

What about the non-diagonalizable case?

For general (non-diagonalizable) linear discrete-time systems, the stability criterion is the same as the diagonalizable case.

General D.T. Linear System:

- Stable iff $|\lambda_i| < 1 \quad \forall i$
- Marginally Stable iff $|\lambda_i| \leq 1$ with at least one $|\lambda_i| = 1$

We will prove (eventually) that all square matrices A can be upper-triangularized

$$TAT^{-1} = U = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ 0 & & & \lambda_n \end{bmatrix}$$

Let $z[k] = T\,x[k]$

$x[k] = T^{-1} z[k]$

$$z[k+1] = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ & & & \lambda_n \end{bmatrix} z[k] + B_{new}\, u[k]$$

Start with "n^{th}" dimension of

$z[k]$

$$z_n[k+1] = \lambda_n z_n[k] + b_n u[k]$$

This sub system is stable

iff $|\lambda_n| < 1$

$|\lambda_n| < 1 \Rightarrow z_n[k]$ bounded

for any bounded input

Now: $n-1$

$$z_{n-1}[k+1] = \lambda_{n-1} z_{n-1}[k] + \underbrace{\left(* \underbrace{z_n[k]}_{\text{bounded}} + b_{n-1} u[k]\right)}_{\hat{u}_{n-1}[k] \text{ bounded}}$$

$|\lambda_{n-1}| < 1 \Leftrightarrow$ sub system is stable/bounded

Repeat for $n-2, \ldots, 1$

$\Rightarrow$ Entire system is stable Iff magnitude of all eigenvalues less than 1.

Recall: "bounded" $u_k := |u_k| < M, \forall k$
$M < \infty$

If $\exists\, i$ such that $|\lambda_i| > 1$

$\Rightarrow$ System is unstable.

Recap: For Linear Discrete-time System

bounded

$$x_{k+1} = A x_k + B u_k + c_k$$

Stable iff all eigenvalues of A have magnitude less than 1.

Continuous-Time

$x(t) \in \mathbb{R}$
$u(t) \in \mathbb{R}$

$$\dot{x}(t) = \lambda x(t) + b u(t)$$

Assume w.l.o.g. $t_0 = 0$

$$x(t) = e^{\lambda t} x(0) + b \int_0^t e^{\lambda(t-s)} u(s)\, ds$$

Assume: $\operatorname{Re}\{\lambda\} > 0$, choose $u(t) \equiv 0 \;\forall t$

$$|x(t)| = |e^{\lambda t} x_0| = |e^{(\alpha + j\omega)t} x_0|$$

$$= |e^{\alpha t} e^{j\omega t} x_0| = \underbrace{|e^{\alpha t}|}_{\to \infty \text{ as } t \to \infty} |x_0|$$

for $Re\{\lambda\} > 0 \Rightarrow$ System Unstable

Assume $Re\{\lambda\} < 0$

$$|x(t)| = \left| e^{\lambda t} x(0) + \int_0^t e^{\lambda(t-s)} b u(s)\, ds \right|$$

$$\leq |e^{\lambda t} x(0)| + \int_0^t |e^{\lambda(t-s)}| \cdot |b| \cdot |u(s)| ds$$

$$\leq |e^{\lambda t} x(0)| + |b| M \int_0^t |e^{\lambda(t-s)}| ds$$

$\lambda = \alpha + j\omega$

$$= |e^{\lambda t} x(0)| + |b| M \int_0^t |e^{\alpha(t-s)}| ds$$

$$\leq |e^{\lambda t} x(0)| + |b| M \left(\frac{e^{\alpha t} - 1}{\alpha} \right)$$

$$\leq |x(0)| + |b| M \left(\frac{e^{\alpha t} - 1}{\alpha} \right)$$

$Re\{\lambda\} < 0 \Rightarrow$ System is stable

$\mathrm{Re}\{\lambda\} = 0 \Rightarrow$ System is marginally stable.

Recap: for scalar linear continuous-time systems:

$\mathrm{Re}\{\lambda\} < 0 \Rightarrow$ Stable

$\mathrm{Re}\{\lambda\} > 0 \Rightarrow$ Unstable

$\mathrm{Re}\{\lambda\} = 0 \Rightarrow$ Marginally Stable

We can show using diagonalization or triangularization

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$x \in \mathbb{R}^n$

$u \in \mathbb{R}^m$

That for vector-valued, linear, continuous-time system:

If $\mathrm{Re}\{\lambda\} < 0$ for all eigenvalues $\Rightarrow$ stable

If $\mathrm{Re}\{\lambda\} > 0$ for <u>any</u> eigenvalue $\Rightarrow$ unstable