Thursday, July 23rd, 2020

- SVD & Least-Squares, continued
- Min-norm control
- System Identification
- Stability (scalar case)

Yesterday:

$$\underset{\substack{n\times m \\ \text{matrix}}}{A} = \underset{n\times n}{U}\,\underset{n\times m}{\Sigma}\,\underset{m\times m}{V^T} \qquad \text{rank}(A) = r$$

$$U = [\underset{n\times r}{U_1} \quad \underset{n\times(n-r)}{U_2}]$$

$$V = [\underset{m\times r}{V_1} \quad \underset{m\times(m-r)}{V_2}]$$

$$\Sigma = \begin{bmatrix} \underset{r\times r}{S} & \underset{r\times(m-r)}{0} \\ \underset{(n-r)\times r}{0} & \underset{(n-r)\times(m-r)}{0} \end{bmatrix}$$

Solution $x^* = \underset{x\in\mathbb{R}^m}{\arg\min} \; \|Ax - y\|_2$

when $y \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times m}$ given, with $n > m$ and A full column-rank

given by $x^* = V S^{-1} U_1^T y = (A^T A)^{-1} A^T y$

Now consider the following

Problem:

$$\boxed{\min_{x \in \mathbb{R}^m} \|x\|_2 \quad \text{such that} \quad y = Ax}$$

when $A \in \mathbb{R}^{n \times m}$ and $y \in \mathbb{R}^n$ are given, $n < m$, and A is full row-rank. (wide Matrix)
rank $= n$

As before, this is equivalent to

$$\boxed{\min_{x \in \mathbb{R}^m} \|x\|_2^2 \quad \text{such that} \quad y = Ax}$$

Let us represent general x using column vectors from V:

$$x = V_1 z_1 + V_2 z_2$$

for some $z_1 \in \mathbb{R}^n$, $z_2 \in \mathbb{R}^{m-n}$

$$\boxed{\min_{\substack{z_1 \in \mathbb{R}^n \\ z_2 \in \mathbb{R}^{m-n}}} \|V_1 z_1 + V_2 z_2\|_2^2 \quad \text{s.t} \quad y = AV_1 z_1 + AV_2 z_2}$$

$$\| V_1 z_1 + V_2 z_2 \|_2^2 = z_1^T V_1^T V_1 z_1$$
$$+ z_2^T V_2^T V_2 z_2$$
$$+ \cancelto{0}{2 z_1^T V_1^T V_2 z_2}$$

$$= \| V_1 z_1 \|_2^2 + \| V_2 z_2 \|_2^2$$

Here I used the fact
that $\| z \|_2^2 = z^T z$
$= |z_1|^2 + |z_2|^2 + |z_3|^2 + \cdots +$

$$y = A V_1 z_1 + \cancelto{0}{A V_2} z_2$$

$$A V_2 = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} V_2$$

$$= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S & 0 \end{bmatrix} \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}$$

$$= 0$$

Equivalently:

$$\min_{\substack{z_1 \in \mathbb{R}^n \\ z_2 \in \mathbb{R}^{m-n}}} \|V_1 z_1\|_2^2 + \|V_2 z_2\|_2^2 \quad \text{s.t.} \quad y = AV_1 z_1$$

$$z_2^* = 0$$

$$y = AV_1 z_1$$

$$= [U_1 \ U_2][S \ 0]\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} V_1 z_1$$

$$= \underset{n \times n}{U} \ \underset{n \times n}{S} \ \underset{n \times 1}{z_1}$$

$$z_1^* = S^{-1} U^T y$$

$$\Rightarrow \quad x^* = V_1 z_1^* + V_2 z_2^*$$

$$= V_1 S^{-1} U^T y$$

Can also show:

$$x^* = A^T (AA^T)^{-1} y$$ (when A full row-rank)

When A is full column-rank:

$$A^{+} := (A^T A)^{-1} A^T$$

When A is full row-rank

$$A^{+} := \left(A^T (A A^T)^{-1} \right)$$

Pseudo-Inverse of A

Optimal Control

Consider a system

$$x_{k+1} = A x_k + B u_k$$

$$\mathcal{C}_k := \begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix}$$

$$\left(x_{goal} - A^k x_0 \right) = \mathcal{C}_k \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$$

$x_k \in \mathbb{R}^n$

$u_k \in \mathbb{R}^m$

$k > n$

Assume controllability

$$\min_{u_0, u_1, \ldots, u_{k-1}} \left\| \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix} \right\|_2$$

$$\text{s.t.} \quad (x_{goal} - A^k x_0) = \mathcal{C}_k \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} u_{k-1}^* \\ \vdots \\ u_0^* \end{bmatrix}}_{k\cdot m} = C_k^T \left(C_k C_k^T \right)^{-1} \left(x_{goal} - A^k x_0 \right)$$

On the homework:

$$\min_{u_0, \dots, u_{k-1}} \sum_{t=0}^{k-1} \| R_t u_t \|_2^2$$

$$\text{s.t.} \quad (x_{goal} - A^k x_0) = C_k \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$$

$$\min_{u_0, \dots, u_{k-1}} \left\| \begin{bmatrix} R_{k-1} & & \\ & \ddots & \\ & & R_0 \end{bmatrix} \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix} \right\|_2^2$$

s.t.

$$(x_{goal} - A^k x_0) = C_k \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$$

This is a least-squares problem

Which looks like

$$\boxed{\begin{array}{ll} \min\limits_{x \in \mathbb{R}^m} & \| Cx \|_2 \\ & \text{s.t.} \quad y = Ax \end{array}}$$

$A \in \mathbb{R}^{n \times m} \qquad n < m$

A full row-rank

Example: Car (without drag)

$$X(t) := \begin{bmatrix} p(t) \\ v(t) \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X(t) \begin{bmatrix} 0 \\ 1/RM \end{bmatrix} u(t)$$

Assume $u(t) = u_k \; : \; t \in [kT, (k+1)T)$

$$V(t) = V(kT) + \int_{kT}^{t} \frac{1}{RM} u_k \quad : \; t \in [kT, (k+1)T]$$

$$= V_k + \frac{(t - kT)}{RM} u_k$$

$$p(t) = p(kT) + \int_{kT}^{t} \left(V_k + \frac{(s - kT)}{RM} \right) ds \quad : \; t \in [kT, (k+1)T]$$

$$V((k+1)T) = V_k + \frac{T}{RM} u_k$$

$$p((k+1)T) = p_k + TV_k + \frac{T^2}{2RM} u_k$$

$\Longrightarrow$

$$X_{k+1} = \underbrace{\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}}_{A} X_k + \underbrace{\frac{1}{RM}\begin{bmatrix} T^2/2 \\ T \end{bmatrix}}_{b} u_k$$

$$X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Lets find u such that

$$X_{goal} = X_k = \begin{bmatrix} P_{goal} \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} b & Ab & \cdots & A^{k-1}b \end{bmatrix}}_{C_k}$$

$$\begin{bmatrix} P_{goal} \\ 0 \end{bmatrix} = C_k \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} b & Ab \end{bmatrix} = \frac{1}{RM}\begin{bmatrix} T^2/2 & 3T^2/2 \\ T & T \end{bmatrix}$$

$\Rightarrow$ Controllable

$$P_{goal} = 1000 \text{ m}$$
$$RM = 500 \text{ kg} \cdot \text{m}$$
$$T = 0.1 \text{ s}$$

$$\begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = C_2^{-1} \begin{bmatrix} 1000 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \times 10^7 \\ -5 \times 10^7 \end{bmatrix} \text{kg} \frac{\text{m}^2}{\text{s}^2}$$

$V_1 = 10000$ m/s

$= 22{,}370$ mph

So lets choose $K > 2$

$$\begin{bmatrix} P_{goal} \\ 0 \end{bmatrix} = C_K \begin{bmatrix} u_{K-1} \\ \vdots \\ \vdots \\ u_0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} u_{K-1}^* \\ \vdots \\ \vdots \\ u_0^* \end{bmatrix} = C_K^T (C_K C_K^T)^{-1} \begin{bmatrix} P_{goal} \\ 0 \end{bmatrix}$$

System ID

How do we learn the parameters of a system model from observations of the system

If we think the system is roughly linear

$$x_{k+1} = \underset{\uparrow}{A} x_k + \underset{\nearrow}{B} u_k$$

learn these

$\Rightarrow$ can use least-squares to solve this problem!

Assume we observe

$$x_0, u_0, x_1, u_1, x_2, \ldots, u_k, x_{k+1}$$

Scalar case:

$$x_k \in \mathbb{R} \qquad u_k \in \mathbb{R}$$

$$x_1 = a x_0 + b u_0 + e_0$$

$$x_2 = a x_1 + b u_1 + e_1$$

$$\vdots$$

$$x_{k+1} = a x_k + b u_k + e_k$$

$$\Updownarrow$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} x_0 & u_0 \\ x_1 & u_1 \\ \vdots & \\ x_k & u_k \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} e_0 \\ \vdots \\ e_k \end{bmatrix}$$

$$\min_{a,b} \left\| \begin{bmatrix} x_1 \\ \vdots \\ x_{k+1} \end{bmatrix} - \begin{bmatrix} x_0 & u_0 \\ \vdots & \\ x_k & u_k \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right\|_2$$

$$\begin{bmatrix} a \\ b \end{bmatrix}^* = \left(\begin{bmatrix} x_0 & u_0 \\ \vdots & \\ x_k & u_k \end{bmatrix}^T \begin{bmatrix} x_0 & u_0 \\ \vdots & \\ x_k & u_k \end{bmatrix} \right)^{-1} \begin{bmatrix} x_0 & u_0 \\ \vdots & \\ x_k & u_k \end{bmatrix}^T \begin{bmatrix} x_1 \\ \vdots \\ x_{k+1} \end{bmatrix}$$

Vector case:

$$x_{k+1} = A x_k + B u_k$$

$$x \in \mathbb{R}^n$$
$$u \in \mathbb{R}^m$$

$$x_{k+1} = A x_k + B u_k$$

$$A := \begin{bmatrix} — & a_1^T & — \\ & \vdots & \\ — & a_n^T & — \end{bmatrix} \qquad B := \begin{bmatrix} — & b_1^T & — \\ & \vdots & \\ — & b_n^T & — \end{bmatrix}$$

$$a_i \in \mathbb{R}^n$$
$$b_i \in \mathbb{R}^m$$

$$x_{k+1} = n\underbrace{\begin{bmatrix} —x_k^T— & & & \\ & —x_k^T— & & \\ & & \ddots & \\ & & & —x_k^T— \end{bmatrix}}_{n^2} \underbrace{\begin{bmatrix} | \\ a_1 \\ | \\ \vdots \\ | \\ a_n \\ | \end{bmatrix}}_{(n^2 \times 1)} + \underbrace{\begin{bmatrix} —u_k^T— & & \\ & \ddots & \\ & & —u_k^T— \end{bmatrix}}_{n \times (n \cdot m)} \underbrace{\begin{bmatrix} | \\ b_1 \\ | \\ \vdots \\ | \\ b_n \\ | \end{bmatrix}}_{\hat{b}}$$

$(n \cdot m \times 1)$

$$x_{k+1} = \begin{bmatrix} - a_1^T - \\ - a_2^T - \\ \vdots \\ - a_n^T - \end{bmatrix} x_k + B u_k$$

$$= \begin{bmatrix} a_1^T x_k \\ a_2^T x_k \\ \vdots \\ a_n^T x_k \end{bmatrix} + B u_k = \begin{bmatrix} x_k^T a_1 \\ \vdots \\ x_k^T a_n \end{bmatrix} + B u_k$$

$$= \underbrace{\begin{bmatrix} - x_k^T - & & & \\ & - x_k^T - & & \\ & & \ddots & \\ & & & - x_k^T - \end{bmatrix}}_{\hat{X}_k} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}_{\hat{a}} + \underbrace{\begin{bmatrix} - u_k^T - & & \\ & \ddots & \\ & & - u_k^T - \end{bmatrix}}_{\hat{U}_k} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{\hat{b}}$$

n

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} \hat{X}_0 & \hat{U}_0 \\ \hat{X}_1 & \hat{U}_1 \\ \vdots & \\ \hat{X}_k & \hat{U}_k \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix}}$$

$n \cdot K > n(n+m)$ if this matrix is to be "tall"

$\Rightarrow K > n+m$

$\llcorner \; n \cdot K \times \underline{(n^2 + n \cdot m)}$