Wednesday, July 22nd, 2020

- Controllability, continued
- Least-squares and SVD
  $\longrightarrow$ optimal control

Controllability:

$$x_{k+1} = A x_k + B u_k$$

$$\left(\begin{array}{l} x(k+1) = Ax(k) + Bu(k) \\ x[k+1] = Ax[k] + Bu[k] \end{array} \right)$$

Assume $x_k \in \mathbb{R}^n$
$u_k \in \mathbb{R}^m$
$A \in \mathbb{R}^{n \times n}$
$B \in \mathbb{R}^{n \times m}$

(Analogous results can be shown for, e.g. $\mathcal{X} = \mathbb{C}^n$, $\mathcal{U} = \mathbb{C}^m$)

System is controllable iff for <u>every</u> $x_{goal} \in \mathbb{R}^n$, there exists a $k \in \{1, 2, \dots\}$ and inputs $u_0, u_1, \dots, u_{k-1}$ such that $x_k = x_{goal}$

Assuming B is a column vector, b, (scalar input), we have

$$x_k - A^k x_0 = \begin{bmatrix} b & Ab & \cdots & A^{k-1}b \end{bmatrix} \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$$

$\Rightarrow$ System is controllable iff $\text{span}\{b, Ab, \dots, A^{k-1}b\} = \mathbb{R}^n$ for some k

Assume $b \neq 0$

$n \geq 2$

check for controllability:

```
k = 1
while true:
    if rank([b Ab ... A^{k-1} b]) == n :
        break
    else:
        k = k+1
```

Situation 1:

$\text{rank}\left(\begin{bmatrix} b & Ab & \cdots & A^{k-1}b \end{bmatrix}\right)$ grows with k up until $k = n$

Situation 2:

$\text{rank}\left(\begin{bmatrix} b & \cdots & A^{k-1}b \end{bmatrix}\right)$ grows up until $k = l < n$ and then the rank stays the same for $k = l+1$

— will rank ever grow again if we continue to increase k?

Lemma:

$$\text{If } \operatorname{rank}(\underbrace{[b \ Ab \ \cdots \ A^{l-1}b]}_{n \times l}) == l$$

$$\text{and } \operatorname{rank}(\underbrace{[b \ Ab \ \cdots \ A^{l} b]}_{n \times (l+1)}) == l$$

$$\text{then } \operatorname{rank}([b \ Ab \ \cdots \ A^{k-1}]) = l$$
$$\forall k \geq l$$

Proof:

$$\operatorname{rank}([b \ Ab \ \cdots \ A^{k-1}b]) = l \quad \forall k \geq l$$

$$\Leftrightarrow \quad A^k b \in \operatorname{Span}\{b, Ab, \cdots, A^{l-1}b\} \ \forall k \geq l$$

$\Leftrightarrow$ There exist scalars $\alpha_0^k, \cdots, \alpha_{l-1}^k$, such that

$$A^k b = \alpha_0^k b + \alpha_1^k Ab + \alpha_2^k A^2 b + \cdots + \alpha_{l-1}^k A^{l-1} b$$

$$\forall k \geq l$$

We will prove this by induction.

Base case:

$n \times \ell \Rightarrow$ full column rank

By assumption of lemma:

$$\operatorname{rank}\left(\underline{[b \ \ Ab \ \cdots \ A^{\ell-1}b]}\right) = \ell$$

and

$$\operatorname{rank}\left([b \ \ Ab \cdots \underline{A^{\ell}b}]\right) = \ell$$

($A^{\ell}b$ is linear combination of $\{b, \dots, A^{\ell-1}b\}$)

Therefore

$$\Rightarrow \underline{A^{\ell}b} = \alpha_0^{\ell} b + \alpha_1^{\ell} Ab + \dots + \alpha_{\ell-1}^{\ell} A^{\ell-1} b$$

for some scalars $\alpha_0^{\ell}, \dots, \alpha_{\ell-1}^{\ell}$

Inductive hypothesis:

"$A^{k}b$ is linear combination of vectors $\{b, \dots, A^{\ell-1}b\}$"

for $k \geq \ell$

$$\underline{A^{k}b} = \alpha_0^{k} b + \alpha_1^{k} Ab + \dots + \alpha_{\ell-1}^{k} A^{\ell-1} b,$$

for some scalars $\alpha_0^{k}, \dots, \alpha_{\ell-1}^{k}$

then:

$$\underline{A^{k+1}b} = A(A^{k}b) = \alpha_0^{k} Ab + \dots + \alpha_{\ell-1}^{k} A^{\ell} b$$

$$= \underbrace{(\alpha_{\ell-1}^{k}\alpha_0^{\ell})}_{\alpha_0^{k+1}} b + \underbrace{(\alpha_{\ell-1}^{k}\alpha_1^{\ell} + \alpha_0^{k})}_{\alpha_1^{k+1}} Ab + \dots + \underbrace{(\alpha_{\ell-1}^{k}\alpha_{\ell-1}^{\ell} + \alpha_{\ell-2}^{k})}_{\alpha_{\ell-1}^{k+1}} A^{\ell-1} b \quad \therefore$$

"$A^{k+1}b$ is also linear combination of $\{b, \dots, A^{\ell-1}b\}$"

We have then that if:

$$\text{rank}\left(\begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}\right) < n$$

$\Rightarrow$ System is uncontrollable

Therefore,

Controllability $\Leftrightarrow$ $\text{rank}\left(\begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}\right) = n$

Multiple inputs:

$$C = \underbrace{\left[\overset{n\times m}{B} \,\middle|\, \overset{n\times m}{AB} \,\middle|\, \cdots \,\middle|\, \overset{n\times m}{A^{n-1}B} \right]}_{n \times (n \cdot m) \text{ matrix}}$$

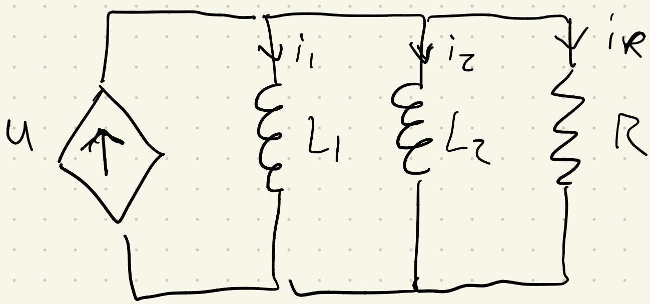
$\text{rank}(C) = n \Leftrightarrow$ System is controllable

Continuous-time systems:

Same as discrete-time

Controllability $\Leftrightarrow$ $\text{rank}(C) = n$

Example: (controllability in continuous time)



$$L_1 \frac{d}{dt} i_1(t) = R\, i_R(t) = R\big(u(t) - i_1(t) - i_2(t)\big)$$

$$L_2 \frac{d}{dt} i_2(t) = R\, i_R(t) = R\big(u(t) - i_1(t) - i_2(t)\big)$$

$$x(t) := \begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} -R/L_1 & -R/L_1 \\ -R/L_2 & -R/L_2 \end{bmatrix} x(t) + \begin{bmatrix} R/L_1 \\ R/L_2 \end{bmatrix} u(t)$$

$$Ab = \begin{bmatrix} -\frac{R}{L_1}\left(\frac{R}{L_1} + \frac{R}{L_2}\right) \\ -\frac{R}{L_2}\left(\frac{R}{L_1} + \frac{R}{L_2}\right) \end{bmatrix} = -\left(\frac{R}{L_1} + \frac{R}{L_2}\right) \cdot b$$

$[b \quad Ab]$ not full rank $\Rightarrow$ uncontrollable

$$L_1 \frac{di_1(t)}{dt} = L_2 \frac{di_2(t)}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(L_1 i_1(t) - L_2 i_2(t)\right) = 0$$

$$L_1 i_1(t) - L_2 i_2(t) = L_1 i_1(0) - L_2 i_2(0)$$

SVD & Least-Squares

Recall: $A = U \Sigma V^T$ $\quad A \in n \times m$

assume $\operatorname{rank}(A) = r \leq \min(n, m)$

$$U = \begin{array}{c c} & \begin{array}{cc} r & n-r \end{array} \\ n & \begin{bmatrix} U_1 & U_2 \end{bmatrix} \end{array}$$

$$V = \begin{array}{c c} & \begin{array}{cc} r & m-r \end{array} \\ m & \begin{bmatrix} V_1 & V_2 \end{bmatrix} \end{array}$$

$$\Sigma = \begin{array}{c c} & \begin{array}{cc} r & m-r \end{array} \\ \begin{array}{c} r \\ n-r \end{array} & \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

Here these "0"s are block matrices of all 0s.

U_1: forms a basis for column space of A

V_2: form a basis for the null space of A

Least-Squares

Recall: $\|x\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}$

Case: A is a tall matrix

$$\begin{array}{cc} & m \\ n & \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix} \end{array}$$

$n > m$, $r = m$

(full column rank)

Assume we have relationship:

$y \in \mathbb{R}^n$
$x \in \mathbb{R}^m$
$e \in \mathbb{R}^n$

$$y = Ax + e$$ ← errors $\neq 0$ if y not in column space of A

Goal:

$$\boxed{\min_x \|e\|_2 \quad \text{such that: } e = y - Ax}$$

Equivalently:

$$\boxed{\min_x \|Ax - y\|_2^2}$$

$$\boxed{\min_x \|Ax - y\|_2}$$

the x^* that solves this problem also solves

$$\boxed{\min_x \|Ax - y\|_2^2}$$

This is because $f(z) = z^2$ is monotonic function on $\mathbb{R}_+$

$(z_1 < z_2 \Leftrightarrow f(z_1) < f(z_2)$ for $z_1, z_2 \in \mathbb{R}_+)$

Because $r = m$

<u>Expanding the objective:</u>

$$\|Ax - y\|_2^2 = \left\| U \begin{bmatrix} S \\ 0 \end{bmatrix} V^T x - y \right\|_2^2$$

Orthonormal Matrices:
$UU^T = I$
$\|Ux\|_2 = \|x\|_2$

"multiply y by UU^T"

$$= \left\| U \left(\begin{bmatrix} S \\ 0 \end{bmatrix} V^T x - U^T y \right) \right\|_2^2$$

$$= \left\| \begin{bmatrix} S \\ 0 \end{bmatrix} V^T x - U^T y \right\|_2^2$$

$$= \left\| \underbrace{\begin{bmatrix} S \\ 0 \end{bmatrix}}_{(m+(n-m)) \times m} \underbrace{V^T}_{m \times m} \underbrace{x}_{m \times 1} - \underbrace{\begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix}}_{(m+(n-m)) \times n} \underbrace{y}_{n \times 1} \right\|_2^2$$

$a \in \mathbb{R}^{n_1}$
$b \in \mathbb{R}^{n_2}$

$$\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\|_2^2 = |a_1|^2 + \dots + |a_{n_1}|^2 + |b_1|^2 + \dots + |b_{n_2}|^2 = \|a\|_2^2 + \|b\|_2^2$$

$$= \left\| \begin{bmatrix} SV^T x - U_1^T y \\ -U_2^T y \end{bmatrix} \right\|_2^2$$

$$= \|SV^T x - U_1^T y\|_2^2 + \underbrace{\|-U_2^T y\|^2}_{\text{independent of } x}$$

There fore:

$$x^* := \min_x \|Ax - y\|_2 = \min_x \|SV^T x - U_1^T y\|_2^2$$

$$\hookrightarrow \boxed{x^* = VS^{-1}U_1^T y}$$

(S and V are both invertible $m \times m$ matrices)

Can also show: $x^* = (A^T A)^{-1} A^T y$ (remember we assumed A full column-rank)

Case: A wide matrix

$$\begin{array}{cc} & m \\ n & \begin{bmatrix} & & & & \end{bmatrix} \end{array}$$

$m > n$

assume full row-rank

Goal:

$$\boxed{\begin{array}{ll} \min & \|x\|_2^2 \\ \text{such that} & y = Ax \end{array}}$$

Tomorrow we will prove that the optimal x^* is given by:

$$x^* = A^T (AA^T)^{-1} y$$

$$= V_1 S^{-1} U^T y$$