Tuesday, July 21st, 2020

How do we use computers to apply controls to continuous-time systems, and how do we analyze the impact of those controls?

- Digital control & discretization
- Change of variables for solving/discretizing diagonalizable systems
- Non-diagonalizable systems?
- Controllability

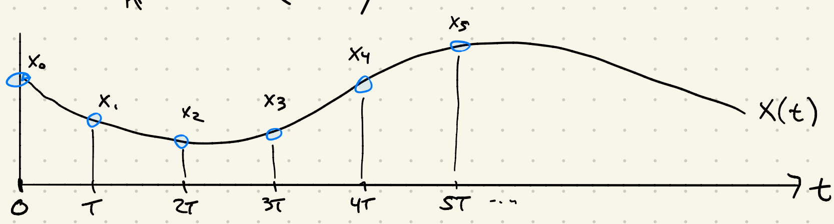
Digital Control:

We wish to control a continuous-time system $\dot{x}(t) = f(x(t), u(t))$ using digital control inputs

↳ $u(t)$ is chosen using samples of $x(t)$ at sampling frequency $\left(\frac{1}{T}\right)$:

$$x(0),\ x(T),\ x(2T),\ x(3T),\ \ldots$$

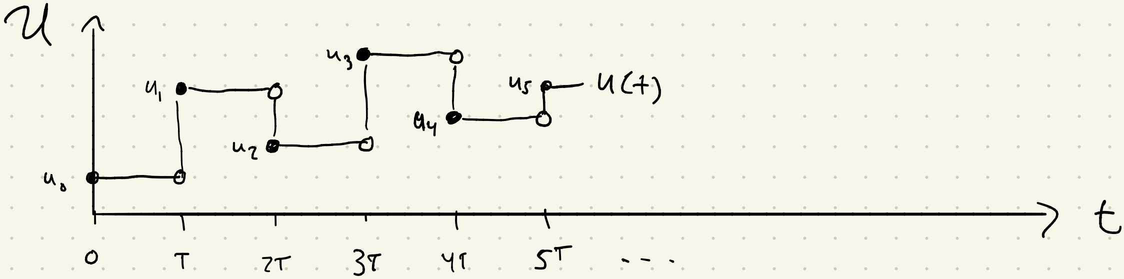
Let $x_k := x(kT)$



We can choose discrete inputs $u_0, u_1, u_2, \ldots,$ to the system

$$u(t) = u_k \quad : \quad t \in [kT, (k+1)T)$$

"Zero-order hold"



Discretization:

Consider Linear System:

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

we wish to derive an equivalent discrete-time system

$$x_{k+1} = A_d\,x_k + B_d\,u_k$$

when $u(t)$ is defined by a zero-order hold

Consider first a scalar system:

$$\dot{x}(t) = \lambda x(t) + b\,u(t)$$

$$x(kT) = x_k$$

$$x(t) = e^{\lambda(t-kT)} x_k + \int_{kT}^{t} e^{\lambda(t-\tau)} b\,u(\tau)\,d\tau$$

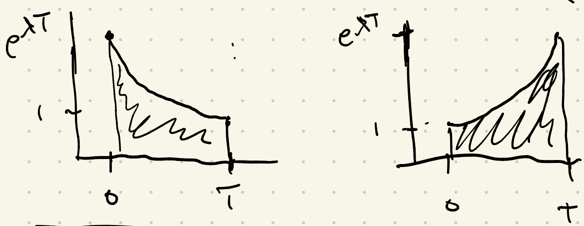
For $t = kT + T$

we have $u(t) = u_k$

$$x(kT+T) = e^{\lambda T} x_k + \int_{kT}^{kT+T} e^{\lambda(kT+T-\tau)}\,d\tau \cdot b\,\underline{u_k}$$

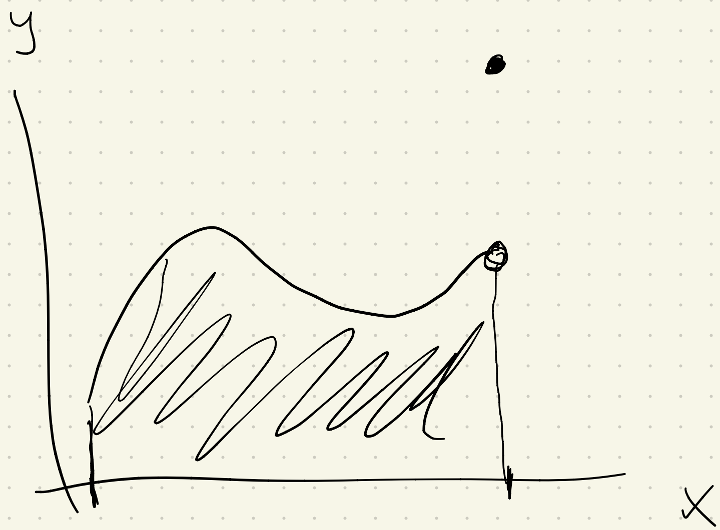
$$\rightarrow \quad = e^{\lambda T} x_k + \int_0^T e^{\lambda(T-\tau)}\,d\tau \cdot b\,u_k$$

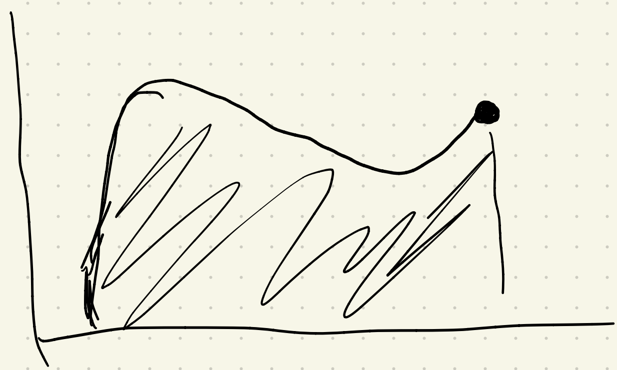
$$\rightarrow \quad = e^{\lambda T} x_k + \underbrace{\int_0^T e^{\lambda s}\,ds}_{} \cdot b\,u_k$$



$$= \begin{cases} \dfrac{e^{\lambda T} - 1}{\lambda} & : \lambda \neq 0 \\[2ex] T & : \lambda = 0 \end{cases}$$

$$x_{k+1} = \underbrace{e^{\lambda T}}_{\lambda_d} x_k + \underbrace{b\left(\int_0^T e^{\lambda s}\,ds\right)}_{b_d} u_k$$


y
x



What about Vector systems?

Consider

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$A := \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad B := \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

i represents i-th dimension of x

$$\rightarrow \quad \dot{x}_i(t) = \lambda_i\, x_i(t) + b_i\, u(t)$$

x at $k+1$ denotes x at time $(k+1)T$

$$\rightarrow x_{k+1} = A_d\, x_k + B_d\, u_k$$

$$A_d := \begin{bmatrix} e^{\lambda_1 T} & & \\ & \ddots & \\ & & e^{\lambda_n T} \end{bmatrix} \quad B_d := \begin{bmatrix} \int_0^T e^{\lambda_1 s}\, ds & & \\ & \ddots & \\ & & \int_0^T e^{\lambda_n s}\, ds \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

<u>Exercise</u>: With diagonal A and general B $(n \times m)$

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \qquad B_d := \begin{bmatrix} \int_0^T e^{\lambda_1 s}\, ds & & \\ & \ddots & \\ & & \int_0^T e^{\lambda_n s}\, ds \end{bmatrix} B$$

What if $\underline{A}$ is not diagonal?

Sometimes it is easier to study the sequence

$$x_0, \quad x_1, \quad x_2, \dots$$

by looking at these variables in a transformed space.

$$z_k := T x_k \qquad x_k := T^{-1} z_k$$

$$z_0, \quad z_1, \quad z_2, \dots,$$

$$x_{k+1} = A_d x_k + B_d u_k$$

$$z_{k+1} = T A_d x_k + T B_d u_k$$

$$z_{k+1} = \underbrace{T A_d T^{-1}}_{A_{new}} z_k + \underbrace{T B_d}_{B_{new}} u_k$$

$$\dot{x}(t) = A x(t) + B u(t)$$

$$\dot{z}(t) = T A T^{-1} z(t) + T B u(t)$$

One useful choice of T is that which diagonalizes A if possible

If A has n lin. independent eigenvectors,

$$\underbrace{A V}_{\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}} = V \underbrace{\Lambda}_{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}$$

$$V^{-1} A V = V^{-1} V \Lambda$$

$$= \underbrace{\Lambda}_{A_{new}}$$

$$T = V^{-1}$$

$$\dot{x}(t) = A x(t) + B u(t)$$

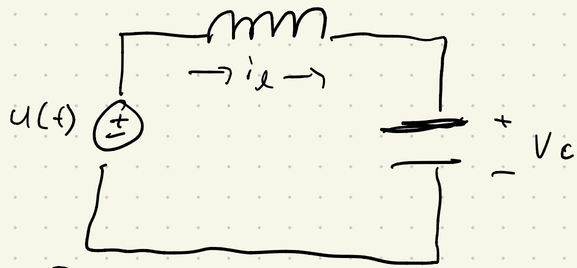
$$\dot{z}(t) = \Lambda z(t) + V^{-1} B u(t)$$

$$z_{k+1} = \begin{bmatrix} e^{\lambda_1 T} & & \\ & \ddots & \\ & & e^{\lambda_n T} \end{bmatrix} z_k + \begin{bmatrix} \int_0^T e^{\lambda_1 s} ds & & \\ & \ddots & \\ & & \int_0^T e^{\lambda_n s} ds \end{bmatrix} V^{-1} B u_k$$

$$x_k = V z_k \qquad z_k = V^{-1} x_k$$

$$x_{k+1} = V \begin{bmatrix} e^{\lambda_1 T} & & \\ & \ddots & \\ & & e^{\lambda_n T} \end{bmatrix} V^{-1} x_k + V \begin{bmatrix} \int_0^T e^{\lambda_1 s} ds & & \\ & \ddots & \\ & & \int_0^T e^{\lambda_n s} ds \end{bmatrix} V^{-1} B u_k$$

<u>Example:</u>



$L = 1$

$C = 1$

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$\lambda_1 = j \quad \Rightarrow \quad v_1 = [1 \ \ -j]^T$$

$$\lambda_2 = -j \qquad v_2 = [1 \ \ j]^T$$

$$V = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \qquad V^{-1} = \frac{1}{2j} \begin{bmatrix} j & -1 \\ j & 1 \end{bmatrix}$$

$$A_d := V \begin{bmatrix} e^{jT} & \\ & e^{-jT} \end{bmatrix} V^{-1}$$

$$= \frac{1}{2j} \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \begin{bmatrix} e^{jT} & \\ & e^{-jT} \end{bmatrix} \begin{bmatrix} j & -1 \\ j & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(e^{jT} + e^{-jT}) & \frac{-1}{2j}(e^{jT} - e^{-jT}) \\ \frac{1}{2j}(e^{jT} - e^{-jT}) & \frac{1}{2}(e^{jT} + e^{-jT}) \end{bmatrix} = \underline{\begin{bmatrix} \cos T & -\sin T \\ \sin T & \cos T \end{bmatrix}}$$

What if A is not diagonalizable?

→ In general this requires solving general linear differential equation $\dot{X}(t) = Ax(t) + Bu(t)$

$\uparrow$ not diagonalizable

$$\dot{X}(t) = Ax(t)$$

$$\Rightarrow \quad X(t) = e^{A(t-t_0)}x(t_0)$$

Controllability of discrete-time systems:

$$X_{k+1} = Ax_k + Bu_k$$

$$\Downarrow$$

$$x_1 = Ax_0 + Bu_0$$

$$x_2 = Ax_1 + Bu_1 = A^2x_0 + ABu_0 + Bu_1$$

$$x_3 = Ax_2 + Bu_2 = A^3x_0 + A^2Bu_0 + ABu_1 + Bu_2$$

$$\vdots$$

$$x_k = A^k x_0 + \begin{bmatrix} B & AB & A^2B & \cdots & A^{k-1}B \end{bmatrix} \begin{bmatrix} u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$$

Definition

choose <u>any</u> $x_{goal} \in \mathcal{X} = \mathbb{R}^n$

Does there exist a sequence of controls $u_0, u_1, \cdots, u_{k-1}$ which brings our system from x_0 to $x_k = x_{goal}$

If so: System <u>Controllable</u>

If not: <u>Uncontrollable</u>

Assume Scalar inputs $u(t)$

$$B \Rightarrow b$$

$$x_k - A^k x_0 = \underbrace{\begin{bmatrix} b & Ab & \cdots & A^{k-1}b \end{bmatrix}}_{\text{column space is } \mathbb{R}^n} \begin{bmatrix} u_{k-1} \\ \vdots \\ \vdots \\ u_0 \end{bmatrix}$$

$k \geq n$ if column space is to be $\mathbb{R}^n$

We will prove tomorrow that $k = n$ is sufficient to check controllability:

Adding additional columns of the form $A^{\ell} b$, $\ell \geq n$ will not increase the dimension of the column space

$$\text{Controllability} \iff \overset{n \times n}{\begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix}} \text{ full rank}$$

$$b: \begin{array}{cc} 1 & \\ \begin{bmatrix} \; \\ \; \\ \; \\ \; \end{bmatrix} & n \end{array}$$

<u>Exercise:</u> think about if

dimension of u $\to$ $m > 1$

Examples:

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$

$$b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad Ab = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\right) = \underline{-1} \Rightarrow \text{full rank}$$

To get to any

x_{goal}

$$\left(x_{goal} - A^2 x_0\right) = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \left(x_{goal} - A^2 x_0\right) = \begin{bmatrix} u_1 \\ u_0 \end{bmatrix}$$

Example:

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k$$

$$x[k+1] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[k]$$

$$\begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^2 b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad A^3 b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x_2[k+1] = 2\, x_2[k]$$

$\uparrow$

evolves independent of u, and $x_1[k]$