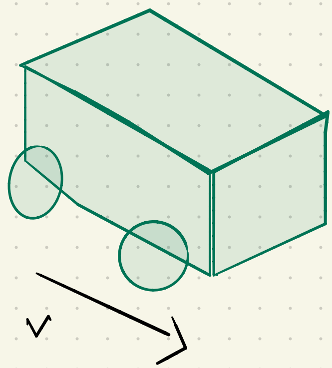
Lecture: Monday, July 20th, 2020

Today:

- Linearization continued
- Solutions to linear/affine differential equations (review)
- Discrete-time systems
- Discretization of continuous-time systems

Last time: Linearization (around equilibria)

Example:



c: drag coef.
a: vehicle area
ρ: air density
M: Mass
R: wheel radius
u(t): wheel torque

$$M\dot{v}(t) = -\frac{1}{2}\rho a c\, v(t)^2 + \frac{1}{R}u(t)$$

$$x(t) := v(t)$$

$$\dot{x}(t) = f(x(t), u(t)) = \frac{-1}{2M}\rho a c\, x(t)^2 + \frac{1}{RM}u(t)$$

$$\mathcal{X} := \mathbb{R}_+ \qquad \mathcal{U} := \mathbb{R}_+$$

$$\mathcal{X}_{eq}(u) := \{ x \in \mathcal{X} \mid f(x,u) = 0 \}$$

$$\mathcal{X}_{eq}(u) := \left\{ + \sqrt{\frac{2u}{R \rho a c}} \right\}$$

$$\Updownarrow$$

$$\mathcal{U}_{eq}(x) = \left\{ \frac{R}{2} \rho a c x^2 \right\}$$

Linearize around $\left(x^*, u^* = \frac{R}{2} \rho a c x^{*2}\right)$

$$\delta x(t) := x(t) - x^*$$

$$\delta u(t) := u(t) - u^*$$

$$\dot{x}(t) = f(x(t), u(t))$$

$$\approx \underbrace{f(x^*, u^*)}_{0 \text{ at eq.}} + \nabla_x f(x^*, u^*)\, \delta x(t) + \nabla_u f(x^*, u^*)\, \delta u(t)$$

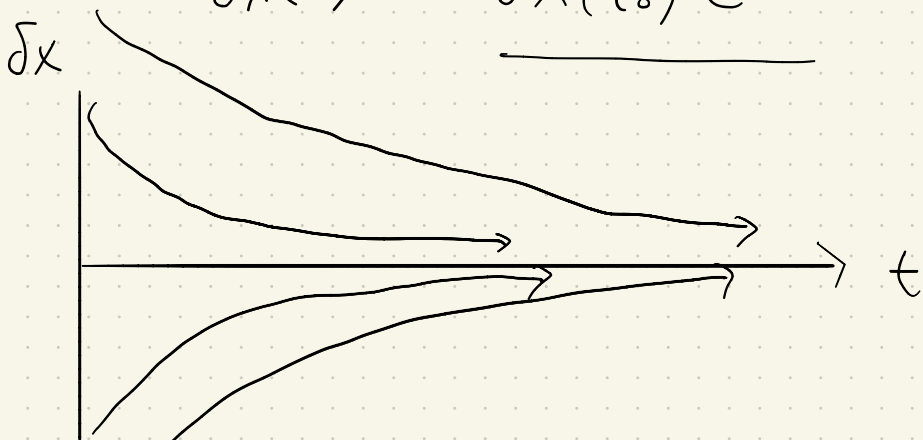
$$\delta \dot{x}(t) = \dot{x}(t) - \cancelto{0}{\frac{d}{dt} x^*} \approx \nabla_x f(x^*, u^*)\, \delta x(t) + \nabla_u f(x^*, u^*)\, \delta u(t)$$

$$\delta\dot{x}(t) = \underset{\substack{\uparrow \\ \nabla_x f(x^*, u^*)}}{\lambda}\, \delta x(t) + \underset{\substack{\uparrow \\ \nabla_u f(x^*, u^*)}}{b}\, \delta u(t)$$

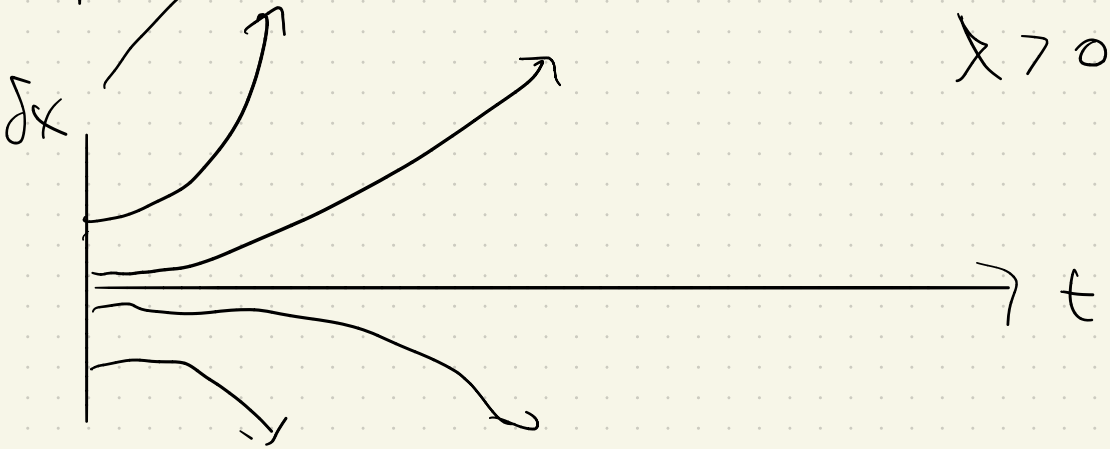
consider $\delta u(t) \equiv 0$

$$\Rightarrow \quad \delta\dot{x}(t) = \lambda\, \delta x(t)$$

$$\delta x(t) = \underline{\delta x(t_0)\, e^{\lambda(t-t_0)}}$$



$\lambda < 0$



$\lambda > 0$

Now consider $\delta u(t) \neq 0$:

Recall:

$$\delta x(t) := e^{\lambda(t-t_0)}\delta x(t_0) + \int_{t_0}^{t} e^{\lambda(t-\tau)} b\,\delta u(\tau)\,d\tau$$

Assume $t_0 = 0$

$$\delta x(t) = e^{\lambda t}\delta x(0) + \int_0^t e^{\lambda(t-\tau)} b\,\delta u(\tau)\,d\tau$$

$\delta u(t) \equiv \delta u \leftarrow$ constant

$$\delta x(t) = e^{\lambda t}\delta x(0) + b\,\delta u \int_0^t e^{\lambda t} e^{-\lambda\tau}\,d\tau$$

$$= e^{\lambda t}\delta x(0) + b\,\delta u\, e^{\lambda t}\left(\frac{-1}{\lambda}\left(e^{-\lambda t} - 1\right)\right)$$

$$= e^{\lambda t}\delta x(0) + \frac{b\,\delta u}{\lambda}\left(e^{\lambda t} - 1\right)$$

Linearization around non-equilibrium

Linearize car around the point

$$(x^*, u^* = 0) \qquad \delta u(t) := u(t) - \cancel{u^*}$$

$$\delta \dot{x}(t) \simeq f(x^*, u^*) + \nabla_x f(x^*, u^*)\, \delta x(t) + \nabla_u f(x^*, u^*)\, \delta u(t)$$

$$= \frac{-1}{2M} \rho a c (x^*)^2 + \left(\frac{-\rho a c}{M} x^*\right) \delta x(t) + \left(\frac{1}{RM}\right) \delta u(t)$$

$$= \lambda\, \delta x(t) + b\, \delta u(t) + c$$

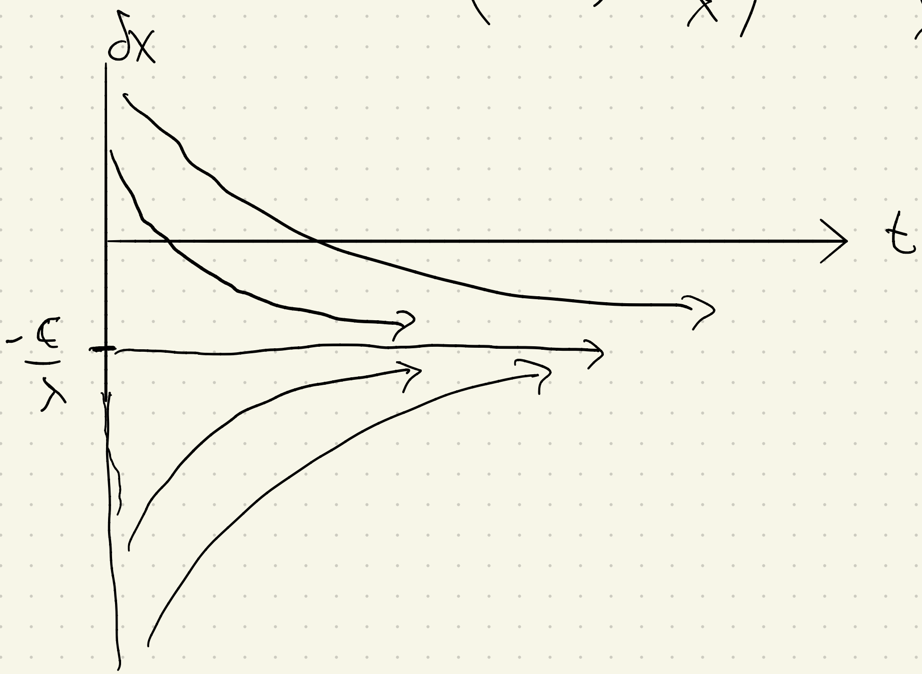
$$\lambda = \left(\frac{-\rho a c}{M} x^*\right) \qquad b = \left(\frac{1}{RM}\right) \qquad c = \left(\frac{-1}{2M} \rho a c\, x^{*2}\right)$$

consider again, $\delta u(t) \equiv 0$

$$\delta \dot{x}(t) \approx \lambda\, \delta x(t) + c$$

$$\delta x(t) = e^{\lambda t}\delta x(0) + \frac{c}{\lambda}\left(e^{\lambda t} - 1\right)$$

$$= e^{\lambda t}\left(\delta x(0) + \frac{c}{\lambda}\right) - \frac{c}{\lambda}$$



Recall for car:

$$-\frac{c}{\lambda} = \frac{\left(-\frac{1}{2m} pac\, x^{*2}\right)}{-\frac{pac}{m} x^{*}} = \frac{-x^{*}}{2}$$

Discrete-time Systems

$$x(t+1) = f(x(t), u(t)), \quad t \in \{0, 1, 2, \dots\}$$

$$x_{t+1} := x(t+1)$$

Equilibrium points:

$$\mathcal{X}_{eq}(u) := \{ x \mid f(x, u) = x \}$$

Linear Systems

$$x_{t+1} = A x_t + B u_t$$

$$\mathcal{X}_{eq}(u) : \{ x \mid (I - A) x = Bu \}$$

Exercise: characterize all the equilibria for a linear discrete-time system.

Example: $s(t)$: inventory of manufacturer @ start of day t

$g(t)$: goods manufactured " " "

$r(t)$: raw material

$u_1(t)$: goods sold

$u_2(t)$: orders placed

$$s(t+1) = s(t) + g(t) - u_1(t)$$
$$g(t+1) = r(t)$$
$$r(t+1) = u_2(t)$$

$$x(t) := \begin{bmatrix} s(t) \\ g(t) \\ r(t) \end{bmatrix} \qquad x(t+1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

$$X_{eq}(u) := \{ x \in \mathbb{R}^3_+ \mid Ax + Bu = x \}$$

Example: (non-linear)

$p(t)$: EECS profs in country

$r(t)$: industry researchers w/ a PhD

$0 < \gamma < 1$: fraction of PhDs become professors

$0 < \alpha < 1$: fraction of field that leave each year

$u(t)$: Number of PhDs that each prof will graduate each year

$$p(t+1) = (1-\alpha)\,p(t) + \gamma\, p(t)\,u(t)$$

$$r(t+1) = (1-\alpha)\,r(t) + (1-\gamma)\,p(t)\,u(t)$$

$$x(t) = \begin{pmatrix} p(t) \\ r(t) \end{pmatrix} \qquad \mathcal{X} := \mathbb{R}_+^2$$

$$\mathcal{U} := \mathbb{R}_+$$

$$\mathcal{X}_{eq}(0) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$\mathcal{X}_{eq}(u) = \left\{ p, r \;\middle|\; \begin{array}{l} (1-\alpha+\gamma u)\,p - p = 0 \\ -\alpha r + (1-\gamma)\,p u = 0 \end{array} \right\}$$

$$u^* = \frac{\alpha}{\gamma} \implies \begin{array}{l} p^* \text{ free} \\ r^* = \dfrac{(1-\gamma)}{\gamma}\, p^* \end{array}$$

Linearization:

$$x(t+1) = f(x(t), u(t))$$
$$\approx f(x^*, u^*) + \nabla_x f(x^*, u^*)\,\delta x(t)$$
$$+ \nabla_u f(x^*, u^*)\,\delta u(t)$$

If x^*, u^* is an equilibrium point:

$$x(t+1) \approx x^* + \nabla_x f(x^*, u^*)\,\delta x(t)$$
$$+ \nabla_u f(x^*, u^*)\,\delta u(t)$$

$$\delta x(t+1) = A\,\delta x(t) + B\,\delta u(t)$$

Exercise: Linearize PhD example around

$$x^* = \begin{pmatrix} p^* \\ \frac{(1-\gamma)}{\gamma} p^* \end{pmatrix}, \quad u^* = \left(\frac{\alpha}{\gamma}\right)$$