Today (July 16^{th}, 2020):

- Recap State-space representation and equilibrium points
- Linearization of non-linear Systems

Yesterday:

We introduced general State model:

$$\begin{cases} \dot{x}_1(t) = f_1(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)) \\ \vdots \\ \dot{x}_n(t) = f_n(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)) \end{cases}$$

or, more compactly:

$$\dot{x}(t) = f(x(t), u(t))$$
$$x(t) \in \mathcal{X}$$
$$u(t) \in \mathcal{U}$$

Typically when written in this form, State and control spaces are defined if not clear from context

Often

$$\mathcal{X} = \mathbb{R}^n$$
$$\mathcal{U} = \mathbb{R}^m$$

When the System is Linear

$$\dot{x}(t) = \underset{n \times n}{A}\, x(t) + \underset{n \times m}{B}\, u(t)$$

Equilibrium Points:

For systems with no inputs

$$\dot{x}(t) = f(x(t))$$

Set of all equilibrium points $\rightarrow$ $$\mathcal{X}_{eq} := \{ x \in \mathcal{X} \mid f(x) = 0 \}$$

For systems with inputs

$$\dot{x}(t) = f(x(t), u(t))$$

Assume a constant input $u \in \mathcal{U}$

$$\mathcal{X}_{eq}(u) := \{ x \in \mathcal{X} \mid f(x, u) = 0 \}$$

For linear systems:

$$\dot{x}(t) = A x(t) + B u(t)$$

$$\mathcal{X}_{eq}(u) := \{ x \in \mathcal{X} \mid Ax + Bu = 0 \}$$

- If A is invertible, for each input u, there is a unique equilibrium point, $x = -A^{-1}Bu$
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- If A is not invertible:
    1. No equilibria exist
    2. If Bu is in the column space of A, there exist a continuum of equilibria for that particular input u.

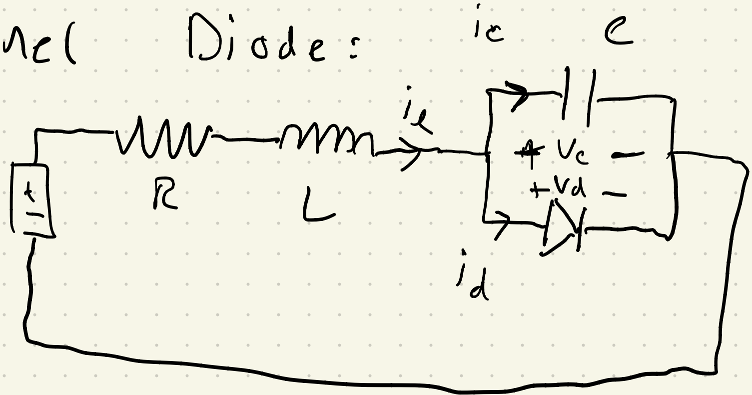
$$A = \underbrace{\begin{bmatrix} | & | & & | & & | \\ u_1 & u_2 & \cdots & u_r & \cdots & u_n \\ | & | & & | & & | \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \cdots 0 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \text{---} & v_1^T & \text{---} \\ & \vdots & \\ \text{---} & v_n^T & \text{---} \end{bmatrix}}_{V^T}$$

First "r" columns of U in SVD of A form orthonormal basis for the column space of A

- Isolated equilibria for linear systems can not exist

* Fully characterize the equilibria for linear systems *

Tunnel Diode:



Tunnel diode:

$i_d = g(V_d)$

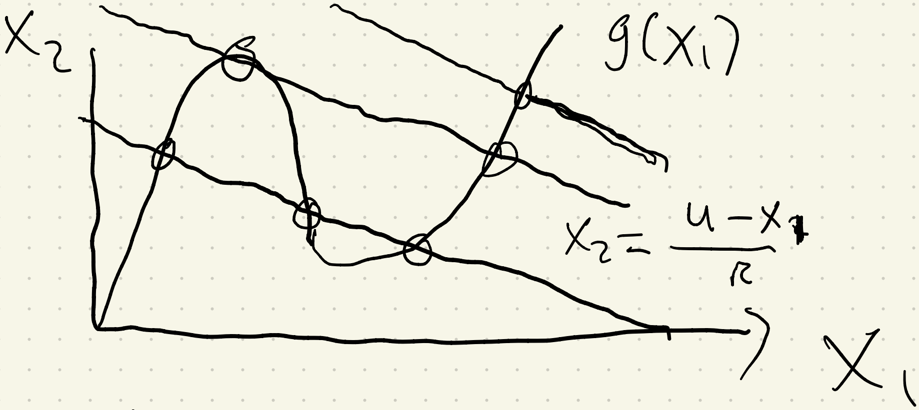
$$C \frac{dV_c(t)}{dt} = i_c(t) = i_\ell(t) - i_d(t) = i_\ell(t) - g(V_c)$$

$$L \frac{di_\ell(t)}{dt} = V_L(t) = -V_c - R i_\ell(t) + V_{in}(t)$$

$$X(t) := \begin{bmatrix} V_c(t) \\ i_\ell(t) \end{bmatrix} \qquad u(t) := V_{in}(t)$$

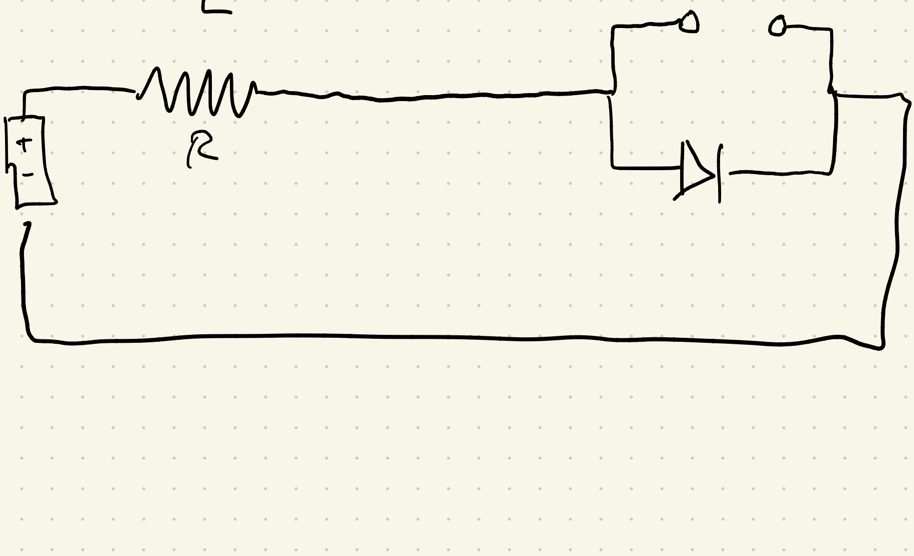
$$\dot{X}(t) = \begin{bmatrix} \frac{x_2(t) - g(x_1(t))}{C} \\ \frac{1}{L}\left(-x_1(t) - R x_2(t) + u(t)\right) \end{bmatrix}$$

$$\mathcal{X}_{eq}(u) := \left\{ X \;\middle|\; \begin{array}{l} x_2 = g(x_1) \\ x_2 = \frac{u - x_1}{R} \end{array} \right\}$$



$$\frac{dV_c}{dt} = \frac{i_c}{C} = 0$$

$$\frac{di_L}{dt} = \frac{V_L}{L} = 0$$



Linearization

So far we have seen that linear systems have some nice properties

- Equilibria can be fully characterized
- Analytic Solutions to linear diferential equations can be found
- easy-to-characterize Stability properties
- easy to design controllers for

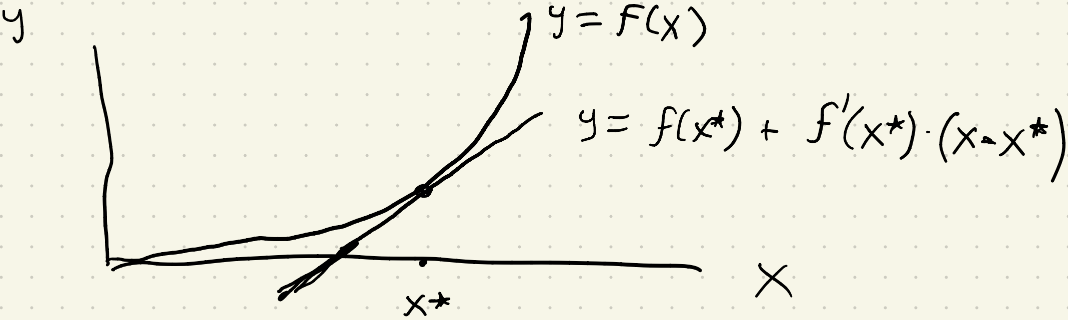
Non-linear systems do not share these properties in general

★ It is very useful to form a linear approximation of a non-linear system around points of interest and analyze the resultant linear system

Recall:

First-order taylor approximation of a funtion $f(x)$

$$f(x) \approx f(x^*) + \nabla f(x^*) \cdot (x - x^*)$$



$$\nabla f := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

"Jacobian Matrix"

$\dot{x}(t) = f(x(t))$, let x^* be a point we wish to linearize around

$$\dot{X}(t) \approx f(x^*) + \nabla f(x^*)\left(X(t) - x^*\right)$$

Define $\quad \delta x(t) := \left(X(t) - x^*\right)$

$$\delta \dot{x}(t) = \dot{x}(t) - \underbrace{\cancel{\dot{x}^*}}_{0} \approx f(x^*) + \nabla f(x^*)\,\delta x(t)$$

$$\delta \dot{x}(t) \approx \underline{\underline{f(x^*)}} + \nabla f(x^*)\,\delta x(t)$$

If x^* is an equilibrium point: $\quad f(x^*) = 0$

$$\delta \dot{x}(t) \approx \underset{\nabla f(x^*)}{\underset{\uparrow}{A}}\, \delta x(t)$$

Now with inputs!

$$\dot{x}(t) = f(x(t), u(t))$$

consider linearization (x^*, u^*)

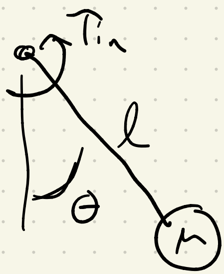
$$\dot{x}(t) \simeq f(x^*, u^*) + \nabla_x f(x^*, u^*)\delta x(t) + \nabla_u f(x^*, u^*)\delta u(t)$$

where $\delta x(t) := (x(t) - x^*)$
$\delta u(t) := (u(t) - u^*)$

$$\nabla_x f := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \leftarrow n \times n$$

$$\nabla_u f := \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix} \leftarrow n \times m$$

$$\delta \dot{x}(t) \simeq \underset{\nabla_x f(x^*, u^*)}{\underset{\uparrow}{A}} \delta x(t) + \underset{\nabla_u f(x^*, u^*)}{\underset{\uparrow}{B}} \delta u(t)$$



$$X(t) = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix} \qquad u(t) = \tau_{in}$$

$$\dot{X}(t) = \begin{bmatrix} x_2(t) \\ -\frac{k}{m} x_2(t) - \frac{g}{\ell} \sin x_1(t) + \frac{u(t)}{m\ell} \end{bmatrix}$$

Two equilibria

<u>downward:</u> $x_1^* = 0$, $x_2^* = 0$, $u^* = 0$

$$\nabla_x f(x^*, u^*) = \begin{bmatrix} 0 & 1 \\ -g/\ell & -k/m \end{bmatrix}$$

$$\nabla_u f(x^*, u^*) = \begin{bmatrix} 0 \\ 1/m\ell \end{bmatrix}$$

<u>upward:</u> $x_1 = \pi$, $x_2 = 0$, $u = 0$

$$\nabla_x f = \begin{bmatrix} 0 & 1 \\ g/\ell & -k/m \end{bmatrix}$$

$$\nabla_u f = \begin{bmatrix} 0 \\ \frac{1}{m\ell} \end{bmatrix}$$

$g = 10$, $\ell = 1$, $m = 1$, $k = 0.1$

Eigenvalues of $\nabla_x F$:

$\lambda = -0.05 \pm 3.162\, i$

$\lambda = \begin{matrix} 3.1127 \\ -3.2127 \end{matrix}$