EE 16B Controls Module

Today:
- Module Overview
- State Space Representation
- Equilibrium Points
- Linearization of non-linear Systems

Module Overview

Study of Systems (physical or Virtual) which have free inputs.

Examples:
- Buildings — HVAC
- cars — Cruise Control
- Airplanes — GPS Flight Path tracking

Questions we should be able to answer at the end of this Module:

Consider a dynamical system:
- how do we Model this system mathematically?
- can we use data to learn parameters of the Model of the system?

- How does the System evolve through time?
- How do we make the System do what we want?
- What is the most efficient way to get it to do what we want?
- Is the system inclined to come to rest? If so, in which configuration?
- Can we change the properties of the system by choice of control?

Preliminaries

$$\dot{x}(t) := \frac{d}{dt} x(t)$$

$$\ddot{x}(t) := \frac{d^2 x(t)}{(dt)^2}$$

$$\vec{x} := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

sometimes we also denote vectors as:

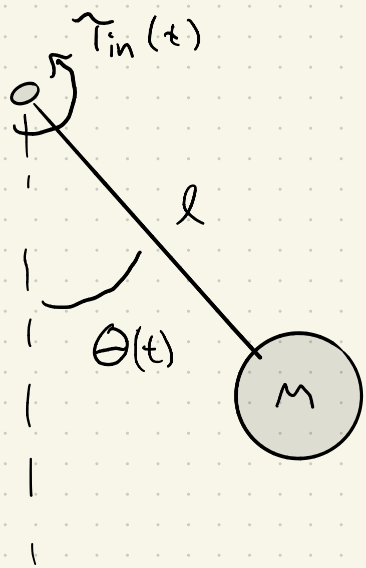
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

State Space Representation

Interested in how systems evolve through time

Use ODEs (ordinary differential equations) to model our systems

Example: Pendulum



$$Ml\,\ddot{\theta}(t) = -Kl\,\dot{\theta}(t) - Mg\sin\theta(t) + \tau_{in}(t)$$

$$X(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} := \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$$

$$\dot{X}(t) = \begin{bmatrix} x_2(t) \\ -g/l \sin(x_1) - \frac{k}{m} x_2(t) + \frac{T_{in}(t)}{ml} \end{bmatrix}$$

$X(t)$ fully represents the state of the system because we can express the time evolution of the system as a function of those variables, constants, and inputs.

$X(t)$ is called the state vector of this system, $x_1(t)$ and $x_2(t)$ are the state variables

$$X(t) \in \mathbb{R}^2, \quad u(t) := T_{in}(t) \in \mathbb{R}$$

For this example,

$\mathbb{R}^2$ is state space

$\mathbb{R}$ is control space

$$\dot{x}(t) = f(\underline{x(t)}, \underline{u(t)})$$

$= \tau_{in}(t)$ for Pendulum

$$f : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{Y}$$

State space

Cartesian Product

Control space

Space of time derivatives

Not always same as $\mathcal{X}$

<u>Pendulum</u>: $\mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$

<u>Definition</u>: Equilibrium point:

$$(x_{eq}, u_{eq}) \in \mathcal{X} \times \mathcal{U}$$

s.t. $f(x_{eq}, u_{eq}) = 0$

Find $\mathbf{X}_{eq}$, τ_{eq}

s.t.

$$\dot{X}(t) = \begin{bmatrix} x_2(t) \\ -g/l \sin x_1(t) - \frac{k}{m} x_2(t) + \frac{\tau_{in}(t)}{ml} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad X_{eq_2} = 0$$

$$\frac{1}{ml}\tau_{eq} = g/l \sin(X_{eq_1})$$

For $\tau_{in}(t) = 0$:

$$\mathbf{X}_{eq} = \{ (n\pi, 0) \mid n \in \mathbb{N} \}$$

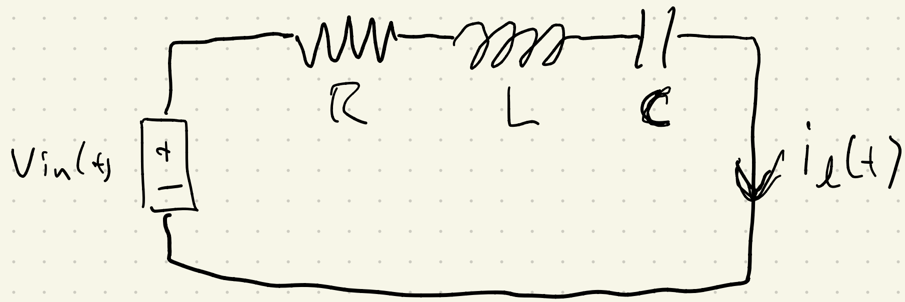
$\theta_{eq} = n\pi$
n is odd

"Unstable equilibrium"

$\theta_{eq} = n\pi$
n even

"Stable" equilibrium

Example: RLC Circuit



$$C \frac{dV_c(t)}{dt} = i_c(t) = i_\ell(t)$$

$$L \frac{di_\ell(t)}{dt} = V_L(t) = -V_c(t) - V_R(t) + V_{in}(t)$$
$$= -V_c(t) - R i_\ell(t) + V_{in}(t)$$

$$X(t) := \begin{bmatrix} V_c(t) \\ i_\ell(t) \end{bmatrix} =: \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\dot{x}(t) = \begin{bmatrix} 1/C \cdot x_2(t) \\ -1/L \cdot x_1(t) - R/L\, x_2(t) + \frac{1}{L} V_{in}(t) \end{bmatrix}$$

$$\dot{X}(t) = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} X(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V_{in}(t)$$

Linear System

Find (X_{eq_1}, u_{eq}) s.t. $\dot{X}_{eq}(t) = 0$

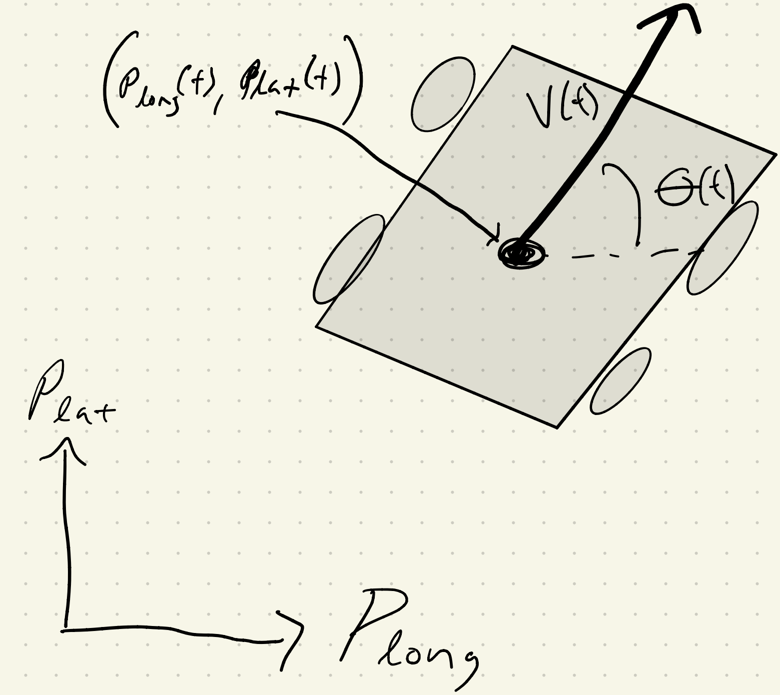
$$\frac{1}{C} X_{eq_2} = 0$$

$$-\frac{1}{L} X_{eq_1} - \frac{R}{L} X_{eq_2} + \frac{1}{L} u_{eq} = 0$$

$\hookrightarrow \quad X_{eq_1} = u_{eq}$

System is at an equilibrium point if $X_1 = V_c = V_{in}$

Example: Car



$$X(t) := \begin{bmatrix} p_{long}(t) \\ p_{lat}(t) \\ \theta(t) \\ V(t) \end{bmatrix} \qquad \dot{X}(t) = \begin{bmatrix} V(t) \cdot \cos\theta(t) \\ V(t) \cdot \sin\theta(t) \\ \omega(t) \\ a(t) \end{bmatrix}$$

$$u(t) := \begin{bmatrix} \omega(t) \\ a(t) \end{bmatrix}$$

Non-linear system

Equilibrium points:

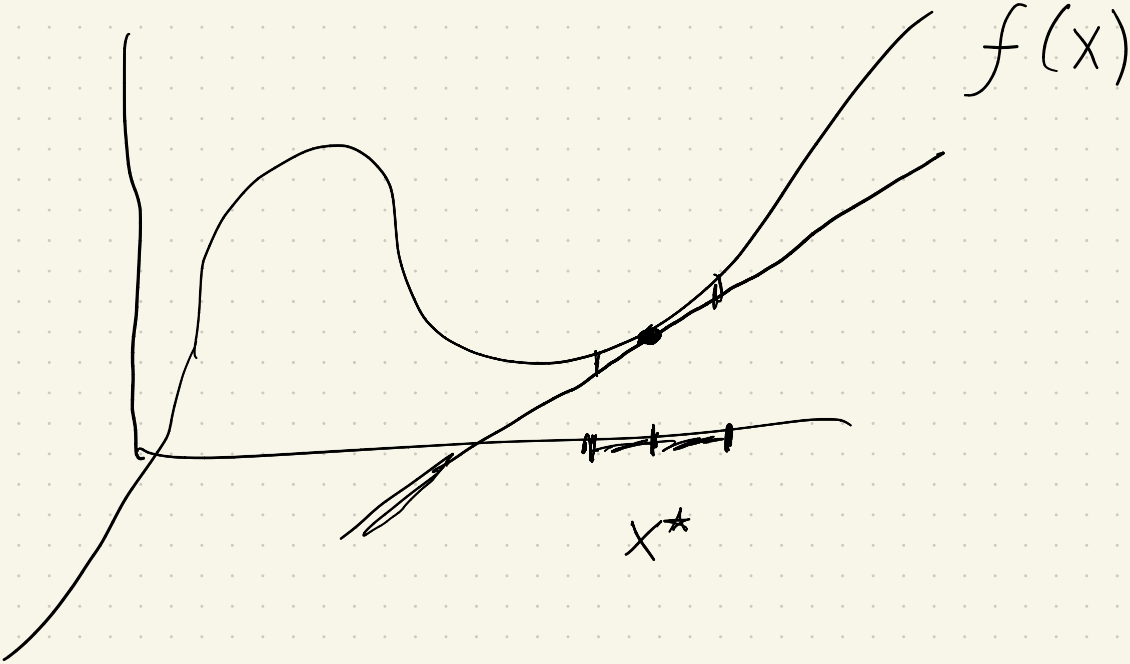
$$X_{eq} := \left\{ \begin{bmatrix} p_{long} \\ p_{lat} \\ \theta \\ 0 \end{bmatrix}, \; \begin{matrix} p_{long} \in \mathbb{R} \\ p_{lat} \in \mathbb{R} \\ \theta \in \mathbb{R} \end{matrix} \right\}$$

$$u_{eq} := \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linear Systems:

$$\dot{x}(t) = A x(t) + B u(t)$$

- Explicit solutions of $x(t)$ (for "integrable" inputs)
- Straightforward analysis of stability
- Easy to design controllers (Stabilizing, Optimal, etc.)
- Can serve as local approximations to non-linear systems



f(x)
x*