This homework is designed as practice for the final and is optional. You will not have to submit anything to Gradescope.

1 Midterm Redos

a) Redo MT1: `http://www.eecs16b.org/exam/mt1.pdf`

Solution

MT1 Solutions

b) Redo MT2: `http://www.eecs16b.org/exam/mt2.pdf`

Solution

MT2 Solutions

2 Stability of State Space Systems (X points)

Consider a discrete time state space system

$$\vec{x}[n+1] = \mathbf{A}\vec{x}[n].$$

For which of the following possible matrices $\mathbf{A}$ is the system stable? Explain your answers.

a)
$$\mathbf{A} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stable? Yes / No
Explanation:

Solution

$\lambda = 0, \frac{1}{2} \implies$ system is stable.

b)
$$\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Stable? Yes / No
Explanation:

Solution

This is a circulant matrix of the signal $x[n] = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{bmatrix}$.

Therefore, its eigenvalues will the $\sqrt{N}$ times the DFT coefficients $X[k]$.

$$x[n] = \frac{1}{2}\cos\left(\frac{2\pi}{4}n\right) \implies X[k] = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_3 = 0 \quad \lambda_4 = 1$$

Since $|\lambda_2| = 1$, the system is unstable.

For parts (c) and (d), consider a continuous time system

$$\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t} = \mathbf{A}\vec{x}(t).$$

c)
$$\mathbf{A} = \begin{bmatrix} -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

Stable? Yes / No
Explanation:

Solution

The matrix A has rank 1 meaning it has eigenvalues of 0. Therefore, the system is unstable.

d) Recall that we are still considering the continuous time system.

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 & -1 \\ -1 & -2 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 1 & 0 & -1 & -2 \end{bmatrix}$$

Stable? Yes / No
Explanation:

Solution

This is a circulant matrix of the signal $x[n] = \begin{bmatrix} -2 & -1 & 0 & 1 \end{bmatrix}$.

Therefore, its eigenvalues will the $\sqrt{N}$ times the DFT coefficients $X[k]$.

$$X[0] = \frac{1}{2}\sum_{n=0}^{3} x[n] = -1$$

$$X[1] = \frac{1}{2}\sum_{n=0}^{3} x[n]e^{-j\frac{\pi}{2}n} = -1 - j$$

$$X[2] = \frac{1}{2}\sum_{n=0}^{3}(-1)^n x[n] = -1$$

$$X[3] = \overline{X[1]} = -1 + j$$

$$\lambda_1 = -2 \quad \lambda_2 = -2 - 2j \quad \lambda_3 = -2 \quad \lambda_4 = -2 + 2j$$

Since all eigenvalues have real part less than 0, the system is stable.

3 Sampling Theorem

Consider the following signal, $x(t)$ defined as,

$$x(t) = \cos(2\pi t) + \sin(4\pi t)$$

a) Find the maximum frequency, $\omega_{\max}$, in radians per second? In Hertz? (From now on, frequencies will refer to radians per second.)

Solution

There are two distinct frequencies in this signal at $\omega = 2\pi$ and $\omega = 4\pi$. Therefore, $\omega_{\max} = 4\pi$ in radians per second, which is $2\,\mathrm{Hz}$.

b) If I sample every T seconds, what is the sampling frequency?

Solution

$\omega_s = \frac{2\pi}{T}$.

c) What is the smallest sampling period T that would result in an imperfect reconstruction?

Solution

From the Nyquist Sampling Theorem, we must sample at $\omega_s > 2\omega_{\max} = 8\pi$ in order to perfectly reconstruct our signal. Therefore $T = \frac{2\pi}{\omega_s} < \frac{1}{4}$ for a perfect reconstruction. Hence the smallest T for which we cannot reconstruct the signal is $T = \frac{1}{4}$.

4 Aliasing

Watch the following video: `https://www.youtube.com/watch?v=jQDjJRYmeWg`.

Assume the video camera running at 30 frames per second. That is to say, the camera takes 30 photos within a second, with the time between photos being constant.

a) Given that the main rotor has 5 blades, list *all* the possible rates at which the main rotor is spinning in revolutions per second assuming no physical limitations.
Hint: Your answer should depend on k where k can be any integer.

Solution

Let the main rotor have a frequency of $\frac{2\pi}{T}$ radians per second. That is to say, it completes one revolution in T seconds, where T is the period of the revolution.
Let $\Delta = \frac{1}{30}s$ be the sample period. Since there are 5 blades, the rotor will look like itself after it finishes a fifth of a revolution, which takes $\frac{T}{5}$ seconds.
This means that, in one Δ, the rotor could have completed $\frac{T}{5}k$ revolutions, where k is an integer. This means that the rotor could be spinning at $\frac{k}{5\Delta}$ revolutions per second, which is $(6 \times k)\,Hz$.

b) Given that the back rotor has 3 blades and completes 2 revolutions in 1 second **in the video**, list *all* the possible rates at which the back rotor is spinning in revolutions per second assuming no physical limitations.
Hint: Your answer should depend on k where k can be any integer.

Solution

The video will always be sampling with sample period $\Delta = \frac{1}{30}s$. This means that the back rotor could have a period of Δ, 2Δ, 3Δ and so on and the camera would not be able to distinguish between them. Moreover, since a third of a rotation looks exactly like a complete rotation, the camera would not be able to tell the difference between periods of the form $k\frac{\Delta}{3}$ where k is an integer.
In the video, we see the back rotor moving roughly at the rate of 2 revolutions per second, which means that, in the video, it has a period of $\frac{1}{2}$ seconds.
This means that, in Δ seconds, it can finish $\frac{k}{3} + 2\Delta$ revolutions, where the $\frac{k}{3}$ revolutions gets hidden by the sample rate. This means that the rotor could be spinning at $\frac{k}{3\Delta} + 2$ revolutions per second, which is $(10k + 2)\,Hz$.