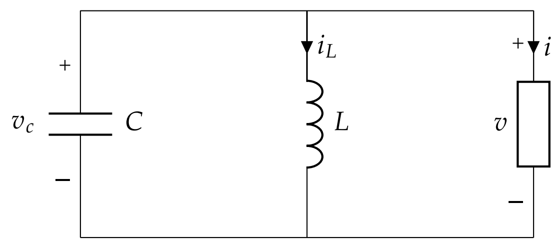
This homework is due on August 4, 2020, at 11:59PM.
Self-grades are due on Tuesday, August 11, 2020, at 11:59PM.

1 Nonlinear circuit component

Note: Solutions to this problem will be released on Wednesday, but you are still required to submit your own original work for this problem.

Consider the circuit below that consists of a capacitor, inductor, and a third element with a nonlinear voltage-current characteristic:

$$i = 2v - v^2 + 4v^3$$



a) Write a state space model of the form

$$\frac{dx_1(t)}{dt} = f_1(x_1(t), x_2(t))$$
$$\frac{dx_2(t)}{dt} = f_2(x_1(t), x_2(t))$$

Where $x_1(t) = v_c(t)$ and $x_2(t) = i_L(t)$.

Solution

We need to get $\frac{dv_c}{dt}$ and $\frac{di_L}{dt}$ in terms of v_c and i_L.
All the components are in parallel, so:

$$v_c = v_L = v$$

Using the relation of an inductor's current and voltage:

$$v_c = L\frac{di_L}{dt}$$
$$\frac{di_L}{dt} = \frac{1}{L}v_c \tag{1}$$

Using KCL, we can say:

$$i_c + i_L + i = 0$$
$$C\frac{dv_c}{dt} + i_L + 2v - v^2 + 4v^3 = 0$$
$$\frac{dv_c}{dt} = \frac{1}{C}\left(-i_L - 2v_c + v_c^2 - 4v_c^3\right) \tag{2}$$

Taking equations (1) and (2) and substituting in x_1 and x_2 gives us our answer:

$$\frac{dx_1}{dt} = f_1(x_1, x_2) = \frac{1}{C}\left(-x_2 - 2x_1 + x_1^2 - 4x_1^3\right)$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2) = \frac{1}{L}x_1$$

b) Linearize the state model at the equilibrium point $x_1 = x_2 = 0$ and specify the resulting A matrix.

Solution

$$A = \nabla f(\vec{x})\Big|_{x_1=x_2=0} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{x_1=x_2=0} = \begin{bmatrix} \frac{1}{C}\left(-2 + 2x_1 - 12x_1^2\right) & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}\Bigg|_{x_1=x_2=0}$$

$$A = \begin{bmatrix} -\frac{2}{C} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}$$

c) Is the linearized system stable?

Solution

$$\det(A - \lambda I) = \lambda^2 + \frac{2}{C}\lambda + \frac{1}{LC} = 0$$

$$\lambda = \frac{-\frac{2}{C} \pm \sqrt{\left(\frac{2}{C}\right)^2 - \frac{4}{LC}}}{2}$$

For a continuous system to be stable, all eigenvalues must have $\mathrm{Re}\{\lambda\} < 0$.

Since both L and C can only take positive values, the square root term will always have a real part smaller than $\frac{2}{C}$, which means both eigenvalues will have negative real parts. The system is stable.

2 Discrete Time Control

Note: Solutions to this problem will be released on Wednesday, but you are still required to submit your own original work for this problem.

Consider the system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 1.5 & 1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)$$

a) Determine if the system is stable.

Solution

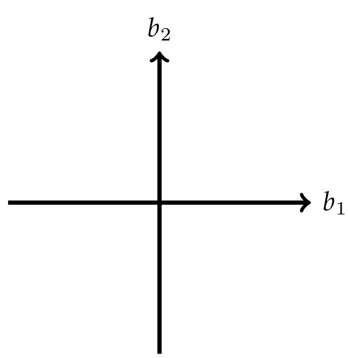
To determine if the system is stable, we need to find the eigenvalues of A.

$$\begin{aligned} \det(\lambda I - A) &= 0 \\ \det(\lambda I - A) &= (\lambda - 1.5)(\lambda - 0.5) - 0 * 1 = (\lambda - 1.5)(\lambda - 0.5) = 0 \\ \lambda_1 &= 0.5 \\ \lambda_2 &= 1.5 \end{aligned}$$

For a discrete system to be stable, $|\lambda| < 1$.

Since $|\lambda_2| > 1$, the sytem is unstable.

b) Determine the set of all (b_1, b_2) values for which the system is **not** controllable and sketch this set of points in the b_1-b_2 plane below.



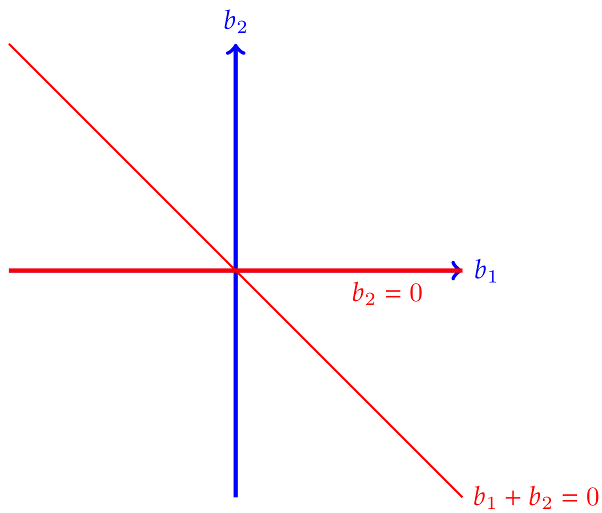
Solution

$$\begin{aligned} C &= \begin{bmatrix} B & AB \end{bmatrix} \\ C &= \begin{bmatrix} b_1 & 1.5b_1 + b_2 \\ b_2 & 0.5b_2 \end{bmatrix} \end{aligned}$$

The system is unstable when C is not full rank. C is not full rank if:

$$\begin{aligned} \frac{1.5b_1 + b_2}{0.5b_2} &= \frac{b_1}{b_2} \\ b_2(1.5b_1 + b_2) - 0.5b_1b_2 &= 0 \\ b_2(b_1 + b_2) &= 0 \end{aligned}$$

The system is not controllable if $b_2 = 0$ or $b_1 + b_2 = 0$.



3 Balance — linearizing a vector system

Note: Solutions to this problem will be released on Wednesday, but you are still required to submit your own original work for this problem.

Justin is working on a small jumping robot named Salto. Salto can bounce around on the ground, but Justin would like Salto to balance on its toe and stand still. In this problem, we'll work on systems that could help Salto balance on its toe using its reaction wheel tail.

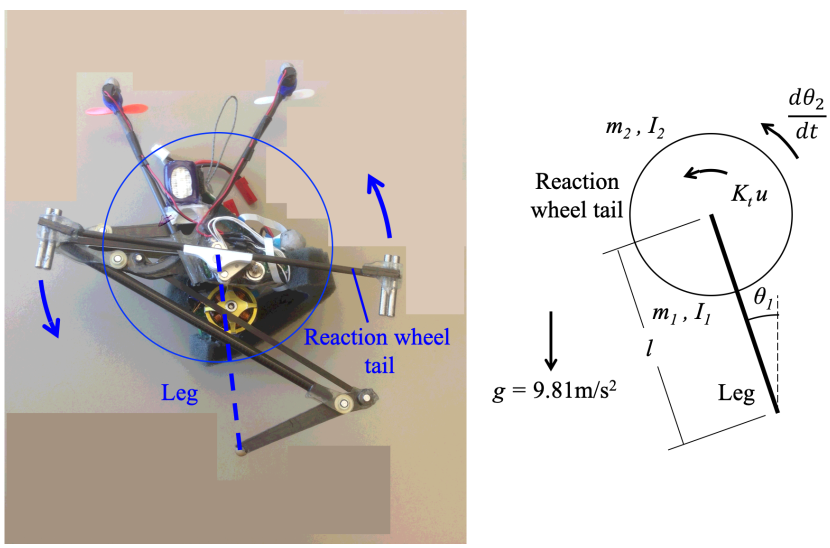


Figure 1: Picture of Salto and the x-z physics model. You can watch a video of Salto here: `https://youtu.be/ZFGxnF9SqDE`

Standing on the ground, Salto's dynamics in the x-z plane (called the sagittal plane in biology) look like an inverted pendulum with a flywheel on the end,

$$(I_1 + (m_1 + m_2)l^2)\frac{\mathrm{d}^2\theta_1(t)}{\mathrm{d}t^2} = -K_t u(t) + (m_1 + m_2)lg\sin\big(\theta_1(t)\big)$$
$$I_2\frac{\mathrm{d}^2\theta_2(t)}{\mathrm{d}t^2} = K_t u(t),$$

where $\theta_1(t)$ is the angle of the robot's body relative to the ground at time t ($\theta_1 = 0\,\mathrm{rad}$ means the body is exactly vertical), $\frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t}$ is the robot body's angular velocity, $\frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t}$ is the angular velocity of the reaction wheel tail, and $u(t)$ is the current input to the tail motor. $m_1, m_2, I_1, I_2, l, K_t$ are positive constants representing system parameters (masses and angular momentums of the body and tail, leg length, and motor torque constant, respectively) and $g = 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}$ is the acceleration due to gravity.

Numerically substituting Salto's physical parameters, the differential equations become:

$$0.001\,\frac{\mathrm{d}^2\theta_1(t)}{\mathrm{d}t^2} = -0.025u(t) + 0.1\sin\bigl(\theta_1(t)\bigr)$$
$$5(10^{-5})\,\frac{\mathrm{d}^2\theta_2(t)}{\mathrm{d}t^2} = 0.025u(t)$$

a) Using the state vector $\begin{bmatrix} \theta_1 \\ \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t} \end{bmatrix}$, and input u, linearize the system about the point $\vec{x}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ with nominal input $u^* = 0$. **Write the linearized equation as $\frac{d}{dt}\vec{x} = A\vec{x} + Bu$. Write out the matrices with the physical numerical values.**

Note: Since the tail is like a wheel, we care only about the tail's angular velocity $\frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t}$ and not its angle $\theta_2(t)$. This is why $\theta_2(t)$ is not a state.

Hint: The sin is the only nonlinearity that you have to deal with here.

Solution

With state vector $\vec{x} = \begin{bmatrix} \theta_1(t) \\ \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t} \end{bmatrix}$, we want to write the dynamics in the form

$$\begin{bmatrix} \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}^2\theta_1(t)}{\mathrm{d}t^2} \\ \frac{\mathrm{d}^2\theta_2(t)}{\mathrm{d}t^2} \end{bmatrix} = A \begin{bmatrix} \theta_1(t) \\ \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t} \end{bmatrix} + Bu(t).$$

From the problem statement, we know that

$$\frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} = \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t}$$
$$\frac{\mathrm{d}^2\theta_1(t)}{\mathrm{d}t^2} = 100\sin\bigl(\theta_1(t)\bigr) - 25u(t)$$
$$\frac{\mathrm{d}^2\theta_2(t)}{\mathrm{d}t^2} = 500u(t).$$

The only nonlinear component is the $\sin\big(\theta_1(t)\big)$ in the $\frac{\mathrm{d}^2\theta_1(t)}{\mathrm{d}t^2} = 100\sin\big(\theta_1(t)\big) - 25u(t)$ equation, which we want to linearize about the point

$$x^* = \begin{bmatrix} \theta_1(t)^* \\ \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t}^* \\ \frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t}^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let's rename this equation as

$$f(\theta_1(t)) = \frac{\mathrm{d}^2\theta_1(t)}{\mathrm{d}t^2} = 100\sin\big(\theta_1(t)\big) - 25u(t)$$

for notational simplicity.

We are finding the tangent line through our operating point $\vec{x}^*$ in order to approximate this function. (When we approximate a function using a line - that is, we linearize it - we can use matrices to represent the function.) We note that the only relevant operating point in this equation is $\theta_1(t)^*$, since this is the only state variable in the equation under consideration. We find the intersect of the line (which is the function evaluated at the operating point, $f(\theta_1(t)^*)$) and the slope of the line (the derivative of the function at the operating point, $f'(\theta_1(t)^*)$) in order to write this linear equation around the operating point. Our linearized equation will then be given by

$$f(\theta_1(t)) = f'(\theta_1(t)^*)\theta_1(t) + f(\theta_1(t)^*).$$

Accordingly,

$$\begin{aligned} f(\theta_1(t)^* = 0) &= 100\sin(0) - 25u(t) \\ &= -25u(t) \\ f(\theta_1(t)^* = 0)' &= \frac{\partial}{\partial\theta_1(t)}(100\sin\big(\theta_1(t)\big) - 25u(t))|_{\theta_1(t)^*} \\ &= 100\cos\big(\theta_1(t)\big)|_{\theta_1(t)^*} \\ &= 100\cos(0) \\ &= 100 \end{aligned}$$

Therefore, the linearized equation is given by

$$f(\theta_1(t)) = \frac{\mathrm{d}^2\theta_1(t)}{\mathrm{d}t^2} = 100\theta_1(t) - 25u(t).$$

Now that all of our equations are linear, we can represent the dynamics with a matrix. Plugging into the matrix form gives:

$$\begin{bmatrix} \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}^2\theta_1(t)}{\mathrm{d}t^2} \\ \frac{\mathrm{d}^2\theta_2(t)}{\mathrm{d}t^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t} \end{bmatrix} + \begin{bmatrix} 0 \\ -25 \\ 500 \end{bmatrix} u(t)$$

b) Your linearized system should have at least one eigenvalue that corresponds to a growing exponential. If we just do the formal test for controllability by checking the $(A, \vec{b})$ pair for the linearized system, **does it indicate that we could place the closed-loop eigenvalues wherever we want for the linearized system?**

Solution

$$C = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & -25 & 0 \\ -25 & 0 & -2500 \\ 500 & 0 & 0 \end{bmatrix}$$

which is full rank, so the system is fully controllable. Because of the proof of CCF, this tells us that we can put the eigenvalues wherever we want for the linearized system. This is because even in continuous time, the closed-loop dynamics are given by $A - BK$ if the control is $u(t) = -Kx(t)$. The CCF proof told us that we can set the characteristic polynomial (and hence the eigenvalues) of $A - BK$ to be whatever we want.

c) Using state feedback, Justin has selected the control gains $K = \begin{bmatrix} 20 & 5 & 0.01 \end{bmatrix}$ for his input $u = K\vec{x}$. **What are the eigenvalues of the closed loop dynamics for the given K? ** Feel free to use numpy.

Solution

With this closed loop feedback, our dynamics become:

$$\begin{bmatrix} \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}^2\theta_1(t)}{\mathrm{d}t^2} \\ \frac{\mathrm{d}^2\theta_2(t)}{\mathrm{d}t^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t} \end{bmatrix} + \begin{bmatrix} 0 \\ -25 \\ 500 \end{bmatrix} \begin{bmatrix} 20 & 5 & 0.01 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ -500 & -125 & -0.25 \\ 10000 & 2500 & 5 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -400 & -125 & -0.25 \\ 10000 & 2500 & 5 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t} \end{bmatrix}$$

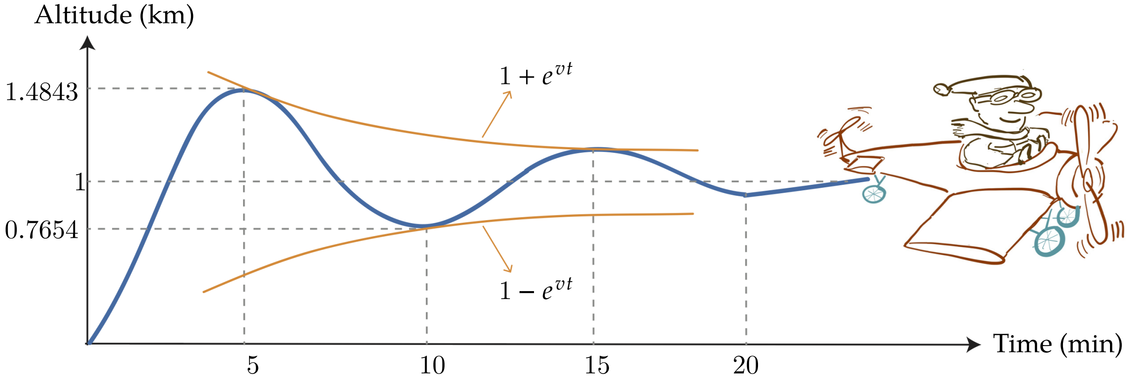
Using numpy to solve, we find the eigenvalues are $\lambda = -116.6$ and $\lambda = -1.697 \pm 1.187j$.

4 Otto the Pilot

Otto has devised a control algorithm, so that his plane climbs to the desired altitude by itself. However, he is having oscillatory transients as shown in the figure. Prof. Arcak told him that if his system has complex eigenvalues

$$\lambda_{1,2} = v \pm j\omega,$$

then his altitude would indeed oscillate with frequency ω about the steady state value, $1\,\mathrm{km}$, and that the time trace of his altitude would be tangent to the curves $1 + e^{vt}$ and $1 - e^{vt}$ near its maxima and minima respectively.



a) Find the real part v and the imaginary part ω from the altitude plot.

Solution

Solving $1 + e^{5v} = 1.4843$ gives us $v = -0.1450\,\frac{1}{\mathrm{min}}$. Then, comparing the maxima that are separated by an interval of 10 minutes gives $\omega = \frac{2\pi}{10} = 0.628\,32\,\frac{\mathrm{rad}}{\mathrm{min}}$.

If you solved in units of $\frac{1}{\mathrm{min}}$ and $\frac{\mathrm{rad}}{\mathrm{s}}$, then $v = -0.0024\,\frac{1}{\mathrm{s}}$ and $\omega = 0.0105\,\frac{\mathrm{rad}}{\mathrm{s}}$.

b) Let the dynamical model for the altitude be

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix},$$

where $y(t)$ is the deviation of the altitude from the steady state value, $\dot{y}(t)$ is the time derivative of $y(t)$, and a_1 and a_2 are constants. Using your answer to part (a), find what a_1 and a_2 are.

Solution

The eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}$ are given by $0 = \lambda^2 - a_2\lambda - a_1$, or equivalently,

$$\lambda = \frac{a_2 \mp \sqrt{a_2^2 + 4a_1}}{2} = v \mp j\omega.$$

Solving for a_1 and a_2 (using the $\frac{1}{\mathrm{min}}$ and $\frac{\mathrm{rad}}{\mathrm{min}}$ values of v and ω), we get

$$a_2 = 2v = -0.2900 \text{ and } a_1 = -\omega^2 - \frac{a_2^2}{4} = -0.4158.$$

If you solved using the $\frac{1}{\text{s}}$ and $\frac{\text{rad}}{\text{s}}$ values of v and ω, then

$$a_2 = 2v = -0.0048 \text{ and } a_1 = -\omega^2 - \frac{a_2^2}{4} = -1.16 \cdot 10^{-4}.$$

c) Otto can change a_2 by turning a knob. Tell him what value he should pick so that he has a "critically damped" ascent with two real negative eigenvalues at the same location.

Solution

To get two real identical eigenvalues, Otto should choose a_2 to make $a_2^2 + 4a_1 = 0$. This means that $a_2 = \pm 2\sqrt{-a_1}$. Since a_2 must be negative for the system to be stable, we only look at the negative root.

Solving with the a_1 derived from the $\frac{1}{\text{min}}$ and $\frac{\text{rad}}{\text{min}}$ values of v and ω, he should tune his knob to

$$a_2 = -2\sqrt{-a_1} = -2\sqrt{0.4158} = -1.2897.$$

If you solved using a_1 derived from the $\frac{1}{\text{s}}$ and $\frac{\text{rad}}{\text{s}}$ values of v and ω, then you get

$$a_2 = -2\sqrt{-a_1} = -2\sqrt{1.16 \cdot 10^{-4}} = -0.0215.$$

5 Stability Analysis: Solving least-squares via gradient descent

In this problem, we will derive a dynamical system approach for solving a least-squares problem which finds the $\vec{x}$ that minimizes $||A\vec{x}-\vec{y}||^2$. We consider A to be tall and full rank.

Recall that the Least Squares problem has a closed-form solution:

$$\vec{\hat{x}} = (A^TA)^{-1}A^T\vec{y}.$$

Direct computation requires the “inversion” of A^TA, which has a complexity of $O(N^3)$ where $(A^TA) \in \mathbb{R}^{N\times N}$. This may be okay for small problems with a few parameters, but can easily become unfeasible for large datasets.

Therefore, we will solve this problem iteratively by the gradient descent method, which we can write as a discrete-time state-space system. The Least-Sqaures problem can be expressed as the following optimization problem

$$\min_{\vec{x}\in\mathbb{R}^n} \left\|\vec{e}\right\|^2 \text{ subject to } \vec{e} = A\vec{x}-\vec{y} \tag{1}$$

a) Given an estimate of $\vec{x}(t)$, we can define the least-squares error $\vec{e}(t)$ to be:

$$\vec{e}(t) = \vec{y} - A\vec{x}(t)$$

Show that if $\vec{x}(t) = \vec{\hat{x}}$, then $\vec{e}(t)$ is orthogonal to the columns of A.

Solution

We can show that $A^T\vec{e}(t) = \vec{0}$. Plugging in $\vec{\hat{x}} = (A^TA)^{-1}A^T\vec{y}$ for $\vec{x}$:

$$\begin{aligned}
\vec{e}(t) &= \vec{y} - A\vec{x}(t)\\
\vec{e}(t) &= \vec{y} - A(A^TA)^{-1}A^T\vec{y}\\
A^T\vec{e}(t) &= A^T\left(\vec{y} - A(A^TA)^{-1}A^T\vec{y}\right)\\
&= A^T\vec{y} - (A^TA)(A^TA)^{-1}A^T\vec{y}\\
&= A^T\vec{y} - IA^T\vec{y}\\
&= A^T\vec{y} - A^T\vec{y}\\
&= \vec{0}
\end{aligned}$$

b) We would like to develop a "fictionary" state space equation for which the state $\vec{x}(t)$ will converge to $\vec{x}(t) \to \vec{\hat{x}}$, the least squares solution. If the argument $A^T(A\vec{\hat{x}}-\vec{y}) = 0$ then we define the following update:

$$\vec{x}(t+1) = \vec{x}(t) - \alpha A^T(A\vec{x}(t)-\vec{y})$$

You can see when $\vec{x}(t) = \vec{\hat{x}}$, the system reaches equilibrium. By the way, it is no coincidence that the gradient of $||A\vec{x}-\vec{y}||^2$ is

$$\nabla||A\vec{x}-\vec{y}||^2 = 2A^T(A\vec{x}-\vec{y})$$

This can be derived by vector derivatives (outside of class scope) or by partial derivatives as we did in the linearization case.

To show that $\vec{x} \to \vec{\hat{x}}$, we define a new state variable $\vec{\epsilon}(t) = \vec{x}(t) - \vec{\hat{x}}$

Derive the discrete-time state evolution equation for $\vec{e}(t)$, and show that it takes the form:

$$\vec{e}(t+1) = (I - \alpha G)\vec{e}(t). \tag{2}$$

Solution

$$\begin{aligned}
\vec{e}(t+1) &= \vec{x}(t+1) - \vec{\hat{x}} \\
&= \vec{x}(t) - \alpha A^T(A\vec{x}(t) - \vec{y}) - \vec{\hat{x}} \\
&= (\vec{x}(t) - \vec{\hat{x}}) - \alpha A^T(A\vec{x}(t) - \vec{y}) \\
&= \vec{e}(t) - \alpha A^T A\vec{x}(t) - \alpha A^T \vec{y} \\
&= \vec{e}(t) - \alpha A^T A(\vec{x}(t) - (A^T A)^{-1} A^T \vec{y}) \\
&= \vec{e}(t) - \alpha A^T A(\vec{x}(t) - \vec{\hat{x}}) \\
&= \vec{e}(t) - \alpha A^T A(\vec{e}(t)) \\
&= (I - \alpha A^T A)\vec{e}(t)
\end{aligned}$$

So $G = A^T A$.

c) We would like to make the system such that $\vec{e}(t)$ converges to 0. **Show that the eigenvalues of matrix $I - \alpha G$ are $1 - \alpha\lambda_{\{G\}}$, where $\lambda_{\{G\}}$ are the eigenvalues of G. Also explain why all of the eigenvalues $\lambda_{\{G\}}$ are greater than zero.**

Solution

Here all we need to notice that if $(\lambda_{\{G\}}, \vec{v})$ is an eigenpair for G, then $(I - \alpha G)\vec{v} = \vec{v} - \alpha\lambda_{\{G\}}\vec{v} = (1 - \alpha\lambda_{\{G\}})\vec{v}$. Hence, the eigenvalues of $I - \alpha G$ are $1 - \alpha\lambda_{\{G\}}$.

The eigenvalues $\lambda_{\{G\}}$ must be greater than zero since $A^T A$ is a matrix of full rank.

d) **For what α would the eigenvalue $1 - \alpha\lambda_{max\{G\}} = 0$ where $\lambda_{max\{G\}}$ is the largest eigenvalue of G. At this α, what would be the largest magnitude eigenvalue of $I - \alpha G$? Is the system stable?**

Solution

When $\alpha = \frac{1}{\lambda_{max\{G\}}}$, $1 - \alpha\lambda_{max\{G\}} = 0$.

To find the largest magnitude eigenvalue of $(I - \alpha G)$, we need to maximize $1 - \alpha\lambda_{\{G\}}$. To remove the least amount, we choose to subtract $\alpha\lambda_{min\{G\}}$ so

$$\lambda_{max\{I-\alpha G\}} = 1 - \frac{\lambda_{min\{G\}}}{\lambda_{max\{G\}}} \tag{3}$$

Since $\lambda_{min\{G\}} > 0$, the discrete-time system will be stable. Furthermore, all the eigenvalues will be in the range $[0, 1)$.

The relationship between the $\lambda_{\{G\}}$ and $\lambda_{I-\alpha G}$ are visually shown on the number lines below.

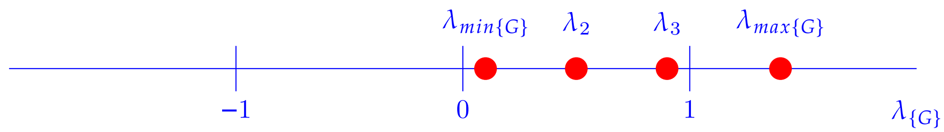


Figure 2: Original eigenvalues of the G matrix. Note all eigenvalues are non-negative. The smallest and largest magnitude eigenvalues are specifically labeled.

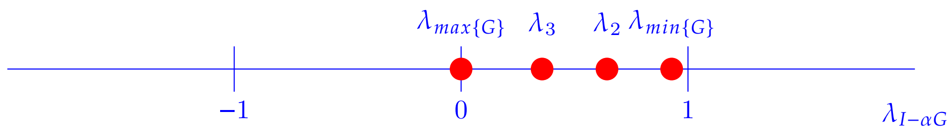


Figure 3: The eigenvalues of $I - \alpha G$ for $\alpha = \frac{1}{\lambda_{max\{G\}}}$. Note that the largest $\lambda_{\{G\}}$ is moved to 0, and the smallest $\lambda_{\{G\}}$ is moved close to 1.

e) We can increase the learning rate α to speed up our convergence. However, we must be careful and not set the learning rate to be too high. **Above what value of α would the system** (2) **become unstable?** You may assume that α is real and positive.

Solution

As we increase the α, the eigenvalues of $(I - \alpha G)$ march to the left. They don't change their order. The system will be unstable when $|1 - \alpha\lambda_{max}| \geq 1$. We can break this down into two cases:

- Case 1: $1 - \alpha\lambda_{max} \geq 1 \implies \alpha\lambda_{max} \leq 0$. Since α and λ_{max} are positive, this is a contradiction.
- Case 2: $1 - \alpha\lambda_{max} \leq -1 \implies \alpha \geq \frac{2}{\lambda_{max}}$.

When $\alpha \geq \frac{2}{\lambda_{max}}$, the set of eigenvalues will cross outside the unit circle.

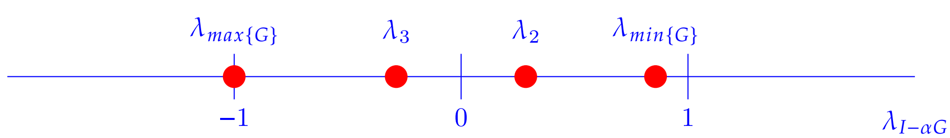


Figure 4: Plot of $\lambda_{I-\alpha G}$ for $\alpha = \frac{2}{\lambda_{max}}$. Note that there is an eigenvalue at -1, so the system is unstable.

6 Controllable Canonical Form and Eigenvalue Placement

Consider a discrete-time linear system below ($\vec{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $B \in \mathbb{R}^n$).

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

If the system is *controllable*, then there exists a transformation $\vec{z} = T\vec{x}$ (where T is the invertible matrix whose columns are the basis vectors for the new representation of the space) such that in the transformed coordinates, the system is in *controllable canonical form*, which is given by

$$\vec{z}(t+1) = \tilde{A}\vec{z}(t) + \tilde{B}u(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \ldots & a_{n-1} \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

Here, $\tilde{A} = TAT^{-1}$ and $\tilde{B} = TB$.
The characteristic polynomials of the matrices A and $\tilde{A}$ are the same and given by

$$\det(\lambda I - A) = \det(\lambda I - \tilde{A}) = \lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \ldots - a_0. \tag{4}$$

a) **Show that A and $\tilde{A}$ have the same eigenvalues.**

Solution

Let λ be an eigenvalue and and $\vec{v}$ its corresponding eigenvector of A. That is, $A\vec{v} = \lambda\vec{v}$. As the hint suggests, we can consider the transformed vector $T\vec{v}$ as a candidate for an equivalent eigenvector for $\tilde{A}$.

Next we evaluate $\tilde{A}T\vec{v}$:

$$\begin{aligned} \tilde{A}(T\vec{v}) &= (TAT^{-1})T\vec{v} \\ &= TA\vec{v} \\ &= T(\lambda\vec{v}) \\ &= \lambda(T\vec{v}) \end{aligned}$$

This shows that λ is indeed an eigenvalue of $\tilde{A}$ and its corresponding eigenvector is $T\vec{v}$.

b) Let the controllability matrices C and $\tilde{C}$ be $C = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ and $\tilde{C} = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix}$, respectively. **Show that the desired transformation is given by $T = \tilde{C}C^{-1}$.**

Solution

$$\begin{aligned} \tilde{C} &= \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \tilde{A}^2\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix} \\ &= \begin{bmatrix} TB & (TAT^{-1})(TB) & (TAT^{-1})(TAT^{-1})(TB) & \cdots & (TAT^{-1})^{n-1}(TB) \end{bmatrix} \\ &= \begin{bmatrix} TB & (TAT^{-1})(TB) & (TA^2T^{-1})(TB) & \cdots & (TA^{n-1}T^{-1})(TB) \end{bmatrix} \\ &= \begin{bmatrix} TB & TAB & TA^2B & \cdots & TA^{n-1}B \end{bmatrix} \\ &= TC \end{aligned}$$

Since by construction the controllability matrix C is full rank and hence invertible,

$$\tilde{C} = TC \tag{5}$$
$$\tilde{C}C^{-1} = TCC^{-1} \tag{6}$$
$$\tilde{C}C^{-1} = T \tag{7}$$

Now, consider the specific **controllable** system

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t) = \begin{bmatrix} -2 & 0 \\ -3 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} u(t) \tag{8}$$

Since the system is controllable, there exists a transformation $\vec{z} = T\vec{x}$ such that

$$\vec{z}(t+1) = \tilde{A}\vec{z}(t) + \tilde{B}u(t) = \begin{bmatrix} 0 & 1 \\ a_0 & a_1 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{9}$$

and the characteristic polynomials of the matrices A and $\tilde{A}$ are the same.

c) **Compute the transformation matrix $T = \tilde{C}C^{-1}$.**

Solution

First, we must calculate the matrix $\tilde{A}$. We know that the characteristic polynomials of A and $\tilde{A}$ are the same, meaning

$$\det(\lambda I - A) = \det(\lambda I - \tilde{A}).$$

Calculating each gives:

$$\det(\lambda I - A) = (\lambda + 2)(\lambda + 1) + 0 = \lambda^2 + 3\lambda + 2$$
$$\det(\lambda I - \tilde{A}) = (\lambda)(\lambda - a_1) - a_0 = \lambda^2 - a_1\lambda - a_0$$
$$\lambda^2 + 3\lambda + 2 = \lambda^2 - a_1\lambda - a_0$$

By inspection, we therefore must have $a_1 = -3$ and $a_0 = -2$. Thus,

$$\tilde{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

Next, we compute the controllability matrices in the original basis and in the controller basis:

$$C = \begin{bmatrix} \frac{1}{2} & -1 \\ \frac{1}{2} & -2 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

Finally, we invert C:

$$C^{-1} = \frac{1}{\frac{1}{2}(-2) - \frac{1}{2}(-1)} \begin{bmatrix} -2 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
$$= -2 \begin{bmatrix} -2 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}$$

Then we have

$$T = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

This systems turns out to be unstable. In the controller basis, we would use a feedback control law $u(t) = -\tilde{K}\vec{z}(t) = \begin{bmatrix} -\tilde{k}_0 & -\tilde{k}_1 \end{bmatrix} \vec{z}(t)$ to place eigenvalues and stabilize the system.

d) If we want to apply the same feedback control law directly using the original $\vec{x}(t)$ state, call the resulting law $u(t) = -K\vec{x}(t) = \begin{bmatrix} -k_0 & -k_1 \end{bmatrix} \vec{x}(t)$. **Give an expression for K in terms of $\tilde{K}$ and T.**

Solution

We are trying to get from the state control law above, $u(t) = -\tilde{K}\vec{z}(t) = \begin{bmatrix} -\tilde{k}_0 & -\tilde{k}_1 \end{bmatrix} \vec{z}(t)$ to a system of the same form involving $\vec{x}(t)$. The beginning of the problem gives us the definition $\vec{z} = T\vec{x}$. So let us substitute and simplify again.

$$u(t) = -\tilde{K}\vec{z}(t) \tag{10}$$
$$= \begin{bmatrix} -\tilde{k}_0 & -\tilde{k}_1 \end{bmatrix} \vec{z}(t) \tag{11}$$
$$= \begin{bmatrix} -\tilde{k}_0 & -\tilde{k}_1 \end{bmatrix} T\vec{x} \tag{12}$$
$$= -K\vec{x} \tag{13}$$

where $K = \tilde{K}T$.

e) **Compute $\tilde{K}$ so that $\tilde{A}_{cl}$ has eigenvalues $\lambda = \pm\frac{1}{2}$ Convert $\tilde{K}$ back to the original basis so that we may obtain our feedback control law for the original $\vec{x}(t)$ state.**

Solution

In the controller basis, we have

$$\begin{aligned} \vec{z}(t+1) &= \tilde{A}\vec{z}(t) + \tilde{B}(-\tilde{K}\vec{z}(t)) \\ &= (\tilde{A} - \tilde{B}\tilde{K})\vec{z}(t) \end{aligned}$$

Plugging in gives

$$\tilde{B}\tilde{K} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} \tilde{k}_0 & \tilde{k}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \tilde{k}_0 & \tilde{k}_1 \end{bmatrix}$$

Thus,

$$\tilde{A}_{cl} = \tilde{A} + \tilde{B}(-\tilde{K}) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\tilde{k}_0 & -\tilde{k}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 - \tilde{k}_0 & -3 - \tilde{k}_1 \end{bmatrix}$$

Since $\tilde{A}_{cl}$ is in controllable canonical form, we know that its bottom row is values $a_{0,cl}$ and $a_{1,cl}$, and that its characteristic polynomial is given by:

$$\det(\lambda I - \tilde{A}_{cl}) = \lambda^2 - a_{1,cl}\lambda - a_{0,cl}$$

Therefore, if we pattern match to the characteristic polynomial with the desired eigenvalues

$$(\lambda + \frac{1}{2})(\lambda - \frac{1}{2}) = \lambda^2 - \frac{1}{4}$$

Then we may easily pick out the appropriate entries of $\tilde{K}$.

$$\begin{aligned} a_{1,cl} &= -3 - \tilde{k}_1 = 0 \\ a_{0,cl} &= -2 - \tilde{k}_0 = -(-\frac{1}{4}) = \frac{1}{4} \end{aligned}$$

Hence,

$$\begin{aligned} \tilde{k}_1 &= -3 \\ \tilde{k}_0 &= -\frac{9}{4} \end{aligned}$$

Converting back to the original basis using the transformation matrix T from before,

$$\begin{aligned} K &= \tilde{K}T \\ &= \begin{bmatrix} -9/4 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -21/4 & -3/4 \end{bmatrix} \end{aligned}$$

Thus,

$$\begin{aligned} k_1 &= -\frac{3}{4} \\ k_0 &= -\frac{21}{4} \end{aligned}$$

7 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

d) **Do you have any feedback on this homework assignment?**

e) **Roughly how many total hours did you work on this homework?**