This homework is due on Tuesday, July 14, 2020, at 11:59PM.
Self-grades are due on Tuesday, July 21, 2020, at 11:59PM.

1 Analyzing Mic Board Circuit

In this problem, we will work up to analyzing a simplified version of the mic board circuit. In lab, we will address the minor differences between the final circuit in this problem and the actual mic board circuit.

The microphone can be modeled as a frequency-dependent current source, $I_{MIC} = k\sin(\omega t) + I_{DC}$, where I_{MIC} is the current generated by the mic (which flows from VDD to VSS), I_{DC} is some constant current, k is the force [1] to current conversion ratio, and ω is the signal's frequency (in $\frac{\text{rad}}{\text{s}}$). VDD and VSS are $5\,\text{V}$ and $-5\,\text{V}$, respectively.

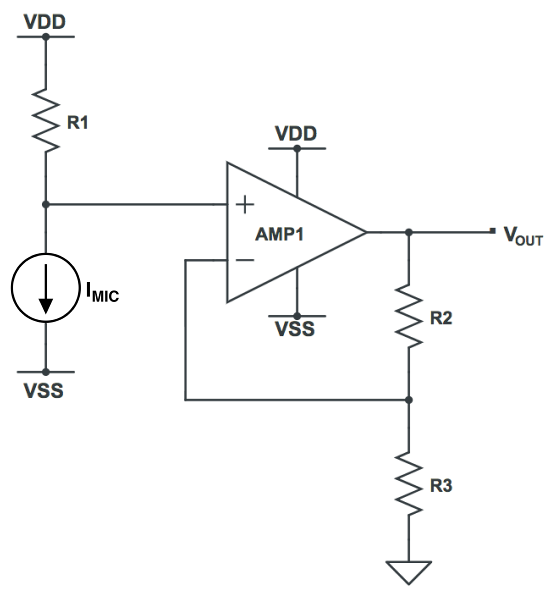


Figure 1: Step 1. The microphone is modeled as a DC current source.

a) **DC Analysis** Assume for now that $k = 0$ (so that we can examine just the "DC" response of the circuit), find V_{OUT} in terms of I_{DC}, R_1, R_2, and R_3 (Hint: You do not need to worry about V_{ss} in your calculations).

Solution

The current in the left branch is equal to I_{DC} since no current flows into the op-amp.

$$V_{in} = V_{DD} - V_{R1} = 5 - (I_{DC} \cdot R_1)$$

$$V_{out} = (1 + \frac{R_2}{R_3}) \cdot V_{in} = (1 + \frac{R_2}{R_3}) \cdot (5 - (I_{MIC} \cdot R_1))$$

b) Now, let's include the sinusoidal part of I_{MIC} as well. We can model this situation as shown below, with I_{MIC} split into two current sources so that we can analyze the whole circuit using superposition. Let $I_{\text{AC}} = k\sin(\omega t)$. Find and plot the function $V_{\text{OUT}(t)}$.

[1] The force is exerted by the soundwaves on the mic's diaphragm.

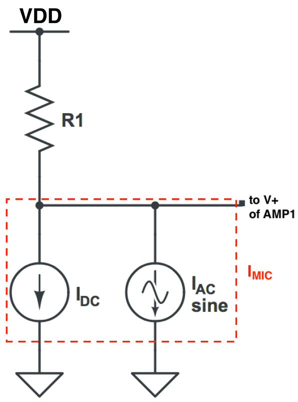


Figure 2: Step 2. The microphone is modeled as the superposition of a a DC and a sinusoidal ("AC") current source.

Solution

Doing superposition, we null each of the sources and add the results. Let's use superposition to find V_{in}. Note, here when we do superposition we have 3 sources that affect V_{in}: V_{DD}, I_{DC}, and I_{AC}. Nulling both current sources, we see that $V_{in_1} = V_{DD}$ because there is no current flowing in our circuit there is no change in voltage over the resistor. Nulling V_{DD} and I_{AC}, we get a similar expression to part (a) except there is no 5 volt source: $V_{in_2} = -R_1 \cdot I_{DC}$. And finally, nulling V_{DD} and I_{DC}, we get a similar expression to our last one: $V_{in_3} = -R_1 \cdot I_{AC}$

Putting these together and plugging in our expression for I_{AC} we get:

$$V_{in} = V_{in_1} + V_{in_2} + V_{in_3} = 5 - R_1 \cdot (ksin(\omega t) + I_{DC})$$

This then goes through a noninverting amplifier for our final answer:

$$V_{out} = (1 + \frac{R_2}{R_3}) \cdot (5 - R_1 \cdot (ksin(\omega t) + I_{DC}))$$

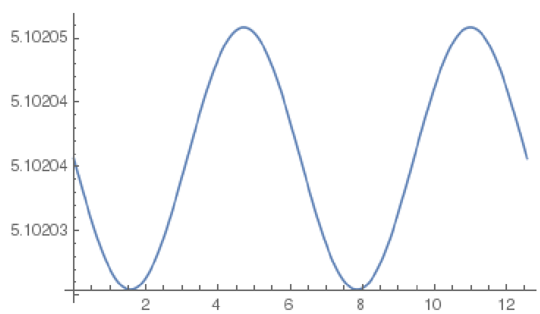


Figure 3: $V_{out}(t)$ when $R_1 = 10\,\mathrm{k\Omega}, R_2 = 2040\,\Omega, R_3 = 100\,\mathrm{k\Omega}, I_{DC} = 10\,\mu\mathrm{A}, k = 10^{-9}$

c) Given that $V_{DD} = 5\,\mathrm{V}$, $V_{SS} = -5\,\mathrm{V}$, $R_1 = 10\,\mathrm{k\Omega}$, and $I_{\mathrm{DC}} = 10\,\mu\mathrm{A}$, find the maximum value of the gain G of the noninverting amplifier circuit for which the op-amp would not need to produce voltages greater than VDD or less than VSS (i.e, find the maximum gain G we can use without causing the op-amp to clip).

Solution

Since the signal is centered around $5 - R_1 I_{DC} = 4.9$V, we know that V_{DD} will limit the amplitude of the signal first.

Using our expression for V_{out} from part (b):

VDD side:

$$G \cdot (5 - R_1 I_{DC} + R_1 \max(k \sin(\omega t)) \leq V_{DD}$$
$$G \cdot (5 - 10^4 \cdot 10^{-5} + 10^4 k)) \leq 5$$
$$G \leq \frac{5}{4.9 + k \cdot 10^4}$$

d) We have modified the circuit as shown below to include a high-pass filter so that the term related to I_{DC} is removed before we apply gain to the signal. Provide a symbolic expression for V_{OUT} given that that $\text{VDD}_0 = 5$ V, $\text{VSS}_0 = -5$ V, $\text{VDD}_1 = 3.3$ V, $\text{VSS}_1 = 0$ V. Show your work.

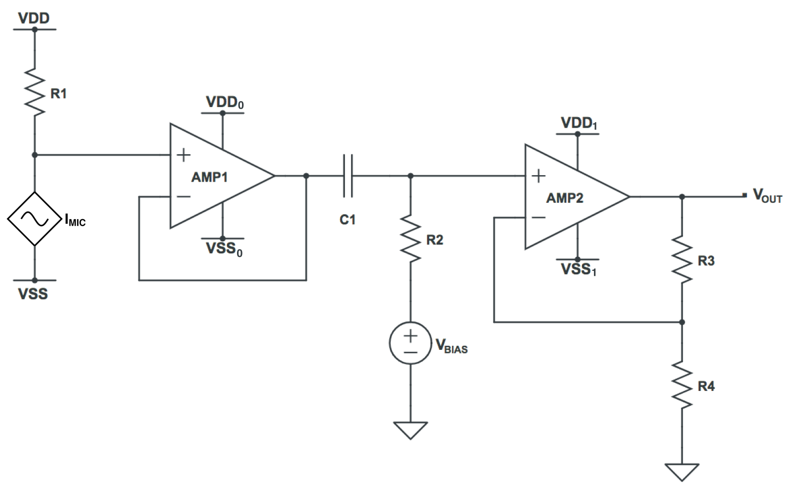


Figure 4: Step 3. Approaching the real mic board circuit. The microphone is still modeled as the superposition of a a DC and a sinusoidal ("AC") current source.

Solution

Since the high-pass filter removes the DC portion of the mic signal (the portion contributed by I_{DC}), the voltage going into the noninverting terminal of AMP2 is $(R_1 k \sin(\omega t) + V_{\text{BIAS}}$, a sinusoid centered around V_{BIAS}. From there, the gain of the noninverting amplifier circuit is $V_{OUT} = (1 + \frac{R_3}{R_4})$, which yields:

$$V_{OUT} = \left(1 + \frac{R_3}{R_4}\right)\left(-R_1 k \sin(\omega t) + V_{\text{BIAS}}\right)$$

e) We would now like to choose V_{BIAS} so that we can get as much gain G out of the non-inverting amplifier circuit (AMP2) as possible without causing AMP2 to clip (i.e, the output of AMP2 must stay between 0V and 3.3V). What value of V_{BIAS} will achieve this goal? If $k = 10^{-5}$ and $R_1 = 10\,\text{k}\Omega$, what is the maximum value of G you can use without having AMP2 clip?

Solution

Since the sinusoidal term has zero mean, we want to put it in the middle of AMP2's range. In other words, we want the output of AMP2 to have a mean of $\frac{3.3-0}{2} = 1.65\,\text{V}$. Since the non-inverting amplifier has a gain of $G = 1 + \frac{R_3}{R_4}$, in order to achieve this we need to choose V_{BIAS} such that $GV_{\text{BIAS}} = 1.65\,\text{V}$.

Therefore, we should choose $V_{\text{BIAS}} = \frac{1.65\,\text{V}}{G}$ as the optimum bias voltage.

Regarding the maximum gain G we can have, consider the output voltage expression:

$$V_{\text{OUT}} = G(-R_1 k \sin(\omega t) + V_{\text{BIAS}})$$

Noting that $GV_{\text{BIAS}} = 1.65\,\text{V}$ by design:

$$3.3V - 1.65V = -GR_1 k \sin(\omega t)$$

$$1.65V = -GR_1 k \sin(\omega t)$$

Letting $\sin(\omega t) = -1$, its maximum value:

$$1.65V = GR_1 k$$

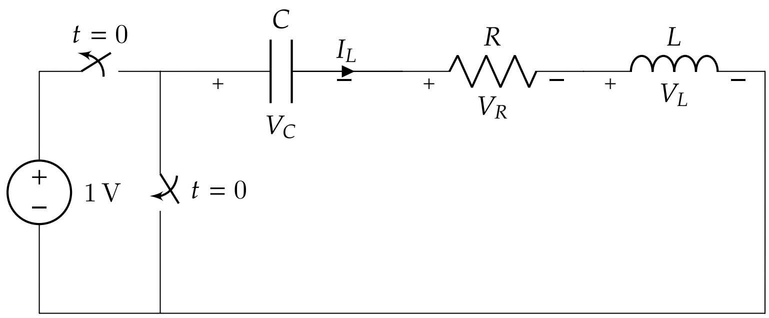
$$1.65V = G(10^4\Omega)(10^{-5}\text{A})$$

$$G = \frac{1.65}{0.1} = 16.5$$

The maximum gain of AMP2 is $G = 16.5$.

2 RLC Responses: Initial Part

Consider the following circuit like you saw in lecture:



Assume the circuit above has reached steady state for $t<0$. At time $t=0$, the switch changes state and disconnects the voltage source, replacing it with a short.

a) **Write a vector differential equations $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$ using state variables $x_1(t) = V_C(t)$ and $x_2(t) = I_L(t)$ that describes this circuit for $t \geq 0$. Remember to find the initial conditions $x_1(0)$ and $x_2(0)$. Also, leave the system symbolic in terms of L, R, and C.**

Solution

We need to find two differential equations, each including a derivative of one of the state variables.

First, let's consider the capacitor equation $I_C(t) = C\frac{d}{dt}V_C(t)$. In this circuit, $I_C(t) = I_L(t)$, so we can write

$$I_C(t) = C\frac{d}{dt}V_C(t) = I_L(t) \tag{1}$$

$$\frac{d}{dt}V_C(t) = \frac{1}{C}I_L(t). \tag{2}$$

If we use the state variable names, we can write this as

$$\frac{d}{dt}x_1(t) = \frac{1}{C}x_2(t), \tag{3}$$

so now we have one differential equation.

For the other differential equation, we can apply KVL around the single loop in this circuit.

$$V_C(t) + V_R(t) + V_L(t) = 0. \tag{4}$$

Using Ohm's Law and the inductor equation $V_L = L\frac{d}{dt}I_L(t)$, we can write this as

$$V_C(t) + RI_L(t) + L\frac{d}{dt}I_L(t) = 0, \tag{5}$$

which we can rewrite as

$$\frac{d}{dt}I_L(t) = -\frac{R}{L}I_L(t) - \frac{1}{L}V_C(t). \tag{6}$$

If we use the state variable names, this becomes

$$\frac{d}{dt}x_2(t) = -\frac{R}{L}x_2(t) - \frac{1}{L}x_1(t), \tag{7}$$

and we have a second differential equation.

To find the intial conditions for $\vec{x}(0)$, if the circuit is in steady state before $t = 0$, then no current is flowing and the capacitor will be fully charged. Therefore:

$$x_1(0) = V_C(0) = 1$$

$$x_2(0) = I_L(0) = 0$$

To summarize the final system is

$$\frac{d}{dt}x_1(t) = \frac{1}{C}x_2(t) \tag{8}$$

$$\frac{d}{dt}x_2(t) = -\frac{1}{L}x_1(t) - \frac{R}{L}x_2(t). \tag{9}$$

which in the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$, will be

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \tag{10}$$

with initial condition

$$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

b) Find the eigenvalues of the A matrix symbolically.

(Hint: the quadratic formula will be involved.)

Solution

To find the eigenvalues, we'll solve $\det(A - \lambda I) = 0$. In other words, we want to find λ such that

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} - \lambda \end{bmatrix}\right) \tag{11}$$

$$= -\lambda\left(-\frac{R}{L} - \lambda\right) + \frac{1}{LC} \tag{12}$$

$$= \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0. \tag{13}$$

The Quadratic Formula gives

$$\lambda = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}. \tag{14}$$

c) Under what condition on the circuit parameters R, L, C are the eigenvalues of A

(i) distinct, and purely real?

(ii) distinct, and purely imaginary?

(iii) distinct, and have nonzero real and imaginary parts?

Solution

(i) For both eigenvalues to be real and distinct, we need the quantity inside the square root to be positive. In other words, we need

$$\frac{R^2}{L^2} - \frac{4}{LC} > 0, \tag{15}$$

or, equivalently,

$$R > 2\sqrt{\frac{L}{C}}. \tag{16}$$

(ii) The only way for both eigenvalues to be purely imaginary is to have $R = 0$. In this case, the eigenvalues would be

$$\lambda = \pm j\sqrt{\frac{1}{LC}}. \tag{17}$$

(iii) For the eigenvalues to have nonzero real and imaginary parts, we need the quantity inside the square root to be negative. In other words, we need

$$R < 2\sqrt{\frac{L}{C}}. \tag{18}$$

d) Now let $R = 1\,\text{k}\Omega$, $L = 25\,\mu\text{H}$, $C = 10\,\text{nF}$ and solve for state variables $x_1(t)$ and $x_2(t)$. Recall from discussion that if λ_1 and λ_2 are the eigenvalues of A, then we can write out our solution as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t} \\ \beta_1 e^{\lambda_1 t} + \beta_2 e^{\lambda_2 t} \end{bmatrix} \tag{19}$$

Feel free to round your answers to two significant figures.

Solution

The first step is to find the eigenvalues of A. By plugging into the formula derived in part (b), we see that

$$\lambda_1 = -1.0 \times 10^5,\ \lambda_2 = -4.0 \times 10^7$$

Using the known form of our solution from discussion, if we plug in our initial condition, we see that

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 \\ \beta_1 + \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Taking the derivative and going back to our original system of differential equations we know that

$$\begin{bmatrix} \frac{d}{dt} x_1(t) \\ \frac{d}{dt} x_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 \alpha_1 e^{\lambda_1 t} + \lambda_2 \alpha_2 e^{\lambda_2 t} \\ \lambda_1 \beta_1 e^{\lambda_1 t} + \lambda_2 \beta_2 e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} \frac{1}{C} x_2(t) \\ -\frac{1}{L} x_1(t) - \frac{R}{L} x_2(t) \end{bmatrix}$$

Plugging in our initial condition, we see that

$$\begin{bmatrix} \frac{d}{dt} x_1(0) \\ \frac{d}{dt} x_2(0) \end{bmatrix} = \begin{bmatrix} \lambda_1 \alpha_1 + \lambda_2 \alpha_2 \\ \lambda_1 \beta_1 + \lambda_2 \beta_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{C} x_2(0) \\ -\frac{1}{L} x_1(0) - \frac{R}{L} x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{L} \end{bmatrix}$$

This gives us the following system of equations

$$\alpha_1 + \alpha_2 = 1 \tag{20}$$
$$\lambda_1\alpha_1 + \lambda_2\alpha_2 = 0 \tag{21}$$
$$\beta_1 + \beta_2 = 0 \tag{22}$$
$$\lambda_1\beta_1 + \lambda_2\beta_2 = \frac{-1}{L} = -4 \cdot 10^4 \tag{23}$$

Solving this system, we get

$$\alpha_1 = 1, \alpha_2 = -0.0025 \tag{24}$$
$$\beta_1 = -0.001, \beta_2 = 0.001 \tag{25}$$

Therefore our solution to the differential equation will be

$$x_1(t) = e^{-1.0\times10^5 t} - 0.0025e^{-4.0\times10^7 t}$$
$$x_2(t) = -0.001e^{-1.0\times10^5 t} + 0.001e^{-4.0\times10^7 t}$$

e) Now let $R = 1\,\Omega, L = 10\,\mu\text{H}, C = 400\,\text{nF}$ and solve for state variables $x_1(t)$ and $x_2(t)$.

Solution

The first step is to find the eigenvalues of A. By plugging into the formula derived in part (b), we see that

$$\lambda_1 = -5\times10^4 + 5\times10^5 j,\ \ \lambda_2 = -5\times10^4 - 5\times10^5 j$$

Using the same approach as the previous part, we can set up the following system of equations

$$\alpha_1 + \alpha_2 = 1 \tag{26}$$
$$\lambda_1\alpha_1 + \lambda_2\alpha_2 = 0 \tag{27}$$
$$\beta_1 + \beta_2 = 0 \tag{28}$$
$$\lambda_1\beta_1 + \lambda_2\beta_2 = \frac{-1}{L} = 4 \cdot 10^7 \tag{29}$$

Solving this system, we get

$$\alpha_1 = 0.5 - 0.05j, \alpha_2 = 0.5 + 0.05j$$
$$\beta_1 = 0.1j, \beta_2 = -0.1j$$

Therefore our solution to the differential equation will be

$$x_1(t) = (0.5 - 0.05j)e^{(-5\times10^4+5\times10^5 j)t} + (0.5 + 0.05j)e^{(-5\times10^4-5\times10^5 j)t} \tag{30}$$
$$= e^{-5\times10^4 t}\cos\left(5\times10^5 t\right) + 0.1e^{-5\times10^4 t}\sin\left(5\times10^5 t\right) \tag{31}$$

$$x_2(t) = 0.1je^{(-5\times10^4+5\times10^5 j)t} - 0.1je^{(-5\times10^4-5\times10^5 j)t} \tag{32}$$
$$= -0.2e^{-5\times10^4 t}\sin\left(5\times10^5 t\right) \tag{33}$$

f) Our answer for the previous part was in terms of complex exponentials. Why did the final voltage and current waveforms end up being purely real?

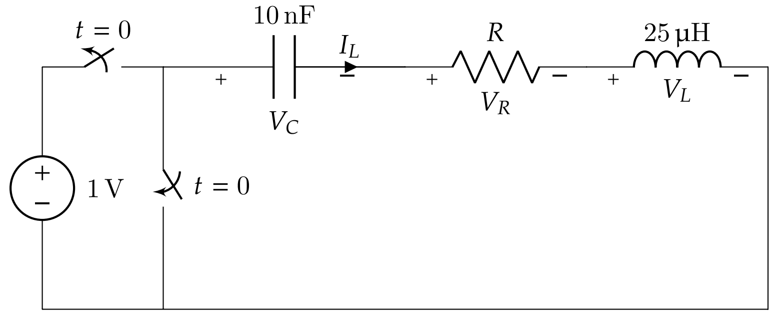
Solution

In this case, it's because the scalars α_1 and α_2 are complex conjugates. Adding these terms will cancel out the imaginary parts due to Euler's identity.

Based on the formula for the eigenvalues in part (b), the two eigenvalues and in fact eigenvectors of an RLC circuit will always be complex conjugates. Also intuitively, since our voltages and observations are always purely real, the solutions modeling the voltage across the capacitior and current across the inductor should also be real-valued.

3 RLC Responses: Critically Damped Case

Building on the previous problem, consider the following circuit with specified component values: (Notice R is not specified yet. You'll have to figure out what that is.)



Assume the circuit above has reached steady state for $t < 0$. At time $t = 0$, the switch changes state and disconnects the voltage source, replacing it with a short.

For this problem, we use the same notations as in Problem 1.

a) For what value of R is there going to be a single eigenvalue of A?

Solution

If the terms under the square root, i.e., the discriminant of the quadratic formula, is 0, then we have a single value. More concretely,

$$\frac{R^2}{L^2} - \frac{4}{LC} = 0 \tag{34}$$

$$\Rightarrow \frac{R^2}{L^2} = \frac{4}{LC} \tag{35}$$

$$\therefore R = 2\sqrt{\frac{L}{C}} = 100\,\Omega \tag{36}$$

b) Find the eigenvalues and eigenspaces of A. What are the dimensions of the corresponding eigenspaces and its implications? (i.e. how many linearly independent eigenvectors can you find associated with this eigenvalue?)

Solution

Our single eigenvalue is,

$$\lambda = -\frac{R}{2L} = -2 \times 10^6 \tag{37}$$

Our system's matrix becomes,

$$A = \begin{bmatrix} 0 & 10^8 \\ -4 \times 10^4 & -4 \times 10^6 \end{bmatrix} \tag{38}$$

Hence, the eigenvector is a basis of the nullspace of $A - \lambda I$,

$$\begin{bmatrix} 2 \times 10^6 & 10^8 \\ -4 \times 10^4 & -2 \times 10^6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{39}$$

This has solutions of the form $\vec{v} = \alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha \begin{bmatrix} -50 \\ 1 \end{bmatrix}$. Since there is only one eigenvector, we have a single dimensional nullspace. This means that we cannot use diagonalization to solve our system of differential equations since we do not have a full basis of eigenvectors.

c) In the provided Jupyter notebook, move the sliders to the resistance value you found in the first part and $C = 10nF$. Sketch $V_c(t)$ and comment on its appearance. Additionally, sketch the location of the eigenvalues on the complex plane. What happens to the voltage and eigenvalues as you slightly increase or decrease R?

Note: You will not be required to turn in the Jupyter Notebook, but you should still run the cells to sketch and comment on the behavior of the circuit.

Solution

At the $R = 100$, $V_c(t)$ appears to decay exponentially. A slight increase in R causes the voltage to decay more slowly. A slight decrease in R causes a voltage undershoot and eventually oscillations. The eigenvalues have converged into the same point at $(-2 \times 10^6, 0)$. Increasing R makes them split into two points, and both points remain on the real axis. One point goes towards the origin, while the other goes towards negative infinity. Decreasing R splits the eigenvalues back into their complex conjugates.

We observe that as we approach the critical value of $R = 100$, the oscillations get damped down faster and faster as we increase R. But then, when we keep going to higher R, the oscillations cease and the plot goes more slowly to 0. The critically damped value for R seems to be in the neighborhood of the fastest drop to near zero.

This can be justified mathematically by showing that the maximum real part of the eigenvalues is minimized at the point where the circuit is critically damped — i.e. at this point, the real part of the eigenvalues is made as negative as possible at this point.

4 Basic Orthonormality Proofs

In this problem, we ask you to establish several important properties of orthonormal bases. Let $U = \begin{bmatrix} \vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{n-1} \end{bmatrix}$ be an n by n matrix, where its columns $\vec{u}_0, \vec{u}_1, \ldots, \vec{u}_{n-1}$ form an orthonormal basis of $\mathbb{C}^n$.

a) **Show that a set of orthonormal vectors $\{\vec{u}_0, \ldots, \vec{u}_{n-1}\}$ must be linearly independent.**

(Hint: Suppose $\vec{w} = \sum_{i=0}^{n-1} \alpha_i \vec{u}_i$, then first show that $\alpha_i = \langle \vec{w}, \vec{u}_i \rangle$. From here ask yourself whether a nonzero linear combination of the $\{\vec{u}_i\}$ could ever be identically zero.)

This basic fact shows how orthogonality is a very nice special case of linear independence.

Solution

Suppose they are not linearly independent, then there exist $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{C}$ such that $\vec{w} = \sum_{i=0}^{n-1} \alpha_i \vec{u}_i = \vec{0}$, while at least one of α_i is non-zero. Meanwhile, for each $\vec{u}_j$, $j \in [0, n-1]$, because $\vec{u}_0, \vec{u}_1, \ldots, \vec{u}_{n-1}$ form an orthonormal basis, we could compute the inner product between $\vec{w}$ and $\vec{u}_j$:

$$\langle \vec{w}, \vec{u}_j \rangle = \sum_{i=0}^{n-1} \alpha_i \langle \vec{u}_i, \vec{u}_j \rangle = \alpha_j \,.$$

Because $\vec{w} = \vec{0}$, α_j should be 0 for all $\langle \vec{u}_j, \vec{w} \rangle$, which is a contradiction to our assumption that at least one of the α_i is non-zero. Therefore, $\vec{u}_0, \ldots, \vec{u}_{n-1}$ are linearly independent.

This confirms what we know — that orthonormality is a particularly robust guarantee of linear independence.

b) **Show that $U^{-1} = U^*$, where U^* is the conjugate transpose of U.**

Solution

By definition, $U^{-1}U = UU^{-1} = I$. If $U^*U = UU^* = I$, then $U^{-1} = U^*$. Let's write down U^* first:

$$U^* = \begin{bmatrix} \vec{u}_0^* \\ \vec{u}_1^* \\ \vdots \\ \vec{u}_{n-1}^* \end{bmatrix}, \tag{40}$$

where $\vec{u}_0^*, \vec{u}_1^*, \ldots, \vec{u}_{n-1}^*$ are the row vectors of U^*. Then, the entry at i-th row and j-th column of U^*U should be $\vec{u}_i^* \vec{u}_j$. Write down the general form for each element of U^*U:

$$(U^*U)_{ij} = \vec{u}_i^* \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

which is the identity matrix, because $\vec{u}_0, \vec{u}_1, \ldots, \vec{u}_{n-1}$ form an orthonormal basis.

Therefore, we conclude by saying $U^*U = I \implies U^* = U^{-1}$.

c) **Show that U preserves inner products, i.e. if $\vec{v}, \vec{w}$ are vectors of length n, then**

$$\langle \vec{v}, \vec{w} \rangle = \langle U\vec{v}, U\vec{w} \rangle \,.$$

Recall that the inner-product is defined to be $\langle \vec{v}, \vec{w} \rangle = \vec{w}^* \vec{v}$.

Solution

For this question, we want to show that:

$$\langle \vec{v}, \vec{w} \rangle = \vec{w}^* \vec{v} = \langle U\vec{v}, U\vec{w} \rangle$$

Using the definition of complex inner products we can write:

$$\langle U\vec{v}, U\vec{w} \rangle = (U\vec{w})^* U\vec{v}$$

Using the form for the complex conjugate of a matrix-vector product as stated in the problem:

$$(U\vec{w})^* U\vec{v} = \vec{w}^* U^* U\vec{v}$$

From the previous problem we know that $U^*U = I$. Therefore:

$$\langle U\vec{v}, U\vec{w} \rangle = \vec{w}^* \vec{v} = \langle \vec{v}, \vec{w} \rangle.$$

This concludes this proof.

d) Let M be a matrix which can be diagonalized by U, i.e. $M = U\Lambda U^*$, where Λ is a diagonal matrix with the eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$ along the diagonal. **Show that M^* has the same set of eigenvectors U, while the eigenvalues of M^* are $\overline{\lambda_0}, \ldots, \overline{\lambda_{n-1}}$.**

Solution

For this problem, we could write down M^* first:

$$M^* = (U\Lambda U^*)^* = U\Lambda^* U^*.$$

This means that Λ^* is the matrix with the eigenvalues $\overline{\lambda_0}, \ldots, \overline{\lambda_{n-1}}$ along the diagonal, while columns of U are eigenvectors of M^*. This implies that M^* has the same eigenvectors as M, with conjugate eigenvalues ($\Lambda^* = \overline{\Lambda}$).

e) Let V be another n by n matrix, where the columns also form an orthonormal basis. **Show that the columns of the product, UV, also form an orthonormal basis.**

Solution

By definition, an orthonormal matrix is a square matrix with orthonormal columns, and its conjugate transpose is equal to its inverse. Such matrices are also called Unitary in the literature, especially when they might be complex. Since V is an orthonormal matrix, we have $V^*V = I$. To show that the columns of UV also form an orthonormal basis, we could write down its conjugate transpose, $(UV)^*$, and apply it to UV:

$$(UV)^*(UV) = V^*U^*UV = V^*V = I\,,$$

which means the columns of UV form an orthonormal basis. This shows that if $W = UV$, and $\vec{w}_i$ is the i^{th} column of W, then

$$\langle \vec{w}_i, \vec{w}_j \rangle = \vec{w}_j^* \vec{w}_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

5 Change of Basis Potpourri

a) Consider a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ represented by an $n \times n$ matrix A that maps $\vec{x} \in \mathbb{R}^n$ to $\vec{y} \in \mathbb{R}^n$:

$$\vec{y} = A\vec{x}$$

Suppose we want to change to another basis for $\mathbb{R}^n$, given by the columns of the matrix P:

$$P = \begin{bmatrix} | & & | \\ \vec{p}_1 & \dots & \vec{p}_n \\ | & & | \end{bmatrix}$$

Then we would like to find the representation $\tilde{A}$ of A in this new basis such that

$$\tilde{A}\tilde{\vec{x}} = \tilde{\vec{y}}$$

where $\tilde{\vec{x}}$ and $\tilde{\vec{y}}$ are the vectors $\vec{x}$ and $\vec{y}$ in the new basis. **Show that $\tilde{A} = P^{-1}AP$.**

Solution

From our knowledge of change of basis, we know that we can convert $\tilde{\vec{x}}$ and $\tilde{\vec{y}}$ back to the old basis via:

$$\vec{x} = P\tilde{\vec{x}}$$
$$\vec{y} = P\tilde{\vec{y}}$$

Substituting into the original equation $A\vec{x} = \vec{y}$:

$$AP\tilde{\vec{x}} = P\tilde{\vec{y}}$$

Mutliplying both sides by P^{-1}

$$P^{-1}AP\tilde{\vec{x}} = \tilde{\vec{y}} = \tilde{A}\tilde{\vec{x}}$$

Pattern matching, we obtain

$$\tilde{A} = P^{-1}AP$$

b) **Suppose we want to transform to an orthonormal basis, so it can be represented by the columns of a unitary matrix U. Show that if A is Hermitian, then $\tilde{A} = U^{-1}AU$ is also Hermitian, i.e. Hermiticity is preserved.**

Solution

Since U is unitary, $U^{-1} = U^*$. Substituting,

$$\tilde{A} = U^*AU$$

Taking the adjoint,

$$\tilde{A}^* = U^*A^*(U^*)^* = U^*A^*U$$

Because A is Hermitian, $A = A^*$. Thus,

$$\tilde{A}^* = U^*AU = \tilde{A}$$

which is the definition of Hermiticity.

c) **Suppose we want to transform to an orthonormal basis, so it can be represented by the columns of a unitary matrix U. Show that if A is unitary, then $\tilde{A} = U^{-1}AU$ is also unitary, i.e. unitarity is preserved.**

Solution

Since U is unitary, $U^{-1} = U^*$. Substituting,

$$\tilde{A} = U^* A U$$

Taking the adjoint,

$$\tilde{A}^* = U^* A^* (U^*)^* = U^* A^* U$$

If $\tilde{A}$ is unitary, then its adjoint should be its inverse:

$$\begin{aligned}\tilde{A}\tilde{A}^* &= U^* A U U^* A^* U \\ &= U^* A A^* U \\ &= U^* U \\ &= I\end{aligned}$$

$$\begin{aligned}\tilde{A}^*\tilde{A} &= U^* A^* U U^* A U \\ &= A^* U U^* A \\ &= A^* A \\ &= I\end{aligned}$$

Thus, $\tilde{A}\tilde{A}^* = \tilde{A}^*\tilde{A} = I$, which is the definition of unitarity.

d) **Prove that the determinant of a matrix A is invariant under change of basis. Further show that for a diagonalizable matrix, the determinant is equal to $\prod_{k=1}^{n} \lambda_k$, where the λ_k are the eigenvalues of A.**

Solution

Note that the determinant of a product is the product of the determinants, i.e.

$$\det AB = \det A \det B$$

Since scalar multiplication is commutative,

$$\det AB = \det A \det B = \det B \det A = \det BA$$

Thus, if we perform a change of basis and transform A to $\tilde{A} = P^{-1}AP$:

$$\begin{aligned}\det \tilde{A} &= \det P^{-1}AP \\ &= \det PP^{-1}A \\ &= \det IA \\ &= \det A\end{aligned}$$

If the basis P is the eigenbasis represented by the matrix V,

$$\begin{aligned}\det A &= \det V^{-1}AV \\ &= \det \Lambda\end{aligned}$$

where the last line comes from the assumption that A is diagonalizable. Λ is a diagonal matrix with the eigenvalues of A making up the diagonal, and since the determinant of a diagonal matrix is the product of the diagonal entries, we conclude:

$$\det A = \prod_{k=1}^{n} \lambda_k$$

e) **Prove that the trace of a matrix A is invariant under change of basis. Further show that for a diagonalizable matrix, the trace is equal to $\sum_{k=1}^{n} \lambda_k$, where the λ_k are the eigenvalues of A.**

As a refresher, the trace of a matrix is defined as

$$Tr(A) = \sum_{i}^{n} A_{ii}$$

i.e., the trace is the sum of the diagonal entries of the matrix.

A helpful property of the trace is that it is *cyclic*:

$$Tr(ABCD) = Tr(DABC) = Tr(CDAB) = Tr(BCDA)$$

Solution

Transforming A to $\tilde{A} = P^{-1}AP$ and using the cyclic property of the trace:

$$\begin{aligned} Tr(\tilde{A}) &= Tr(P^{-1}AP) \\ &= Tr(PP^{-1}A) \\ &= Tr(IA) \\ &= Tr(A) \end{aligned}$$

If the basis P is the eigenbasis represented by the matrix V,

$$\begin{aligned} Tr(A) &= Tr(V^{-1}AV) \\ &= Tr(\Lambda) \end{aligned}$$

where the last line again comes from the assumption that A is diagonalizable. Λ is a diagonal matrix with the eigenvalues of A making up the diagonal, and thus we conclude:

$$Tr(A) = \sum_{k=1}^{n} \lambda_k$$

6 SVD of the derivative operator

The continuous functions from the Real interval $[0, 2\pi)$ to $\mathbb{C}$ are a vector space. In this problem, fix a natural number N and a vector subspace

$$V = \text{Span}\{f_n\}_{n=-N,\ldots,N}, \ (2N+1 \text{ in total})$$

where f_n is the function defined by

$$f_n(\theta) = e^{jn\theta}.$$

Define an inner product by

$$\langle u, v\rangle = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\overline{v(\theta)}\,\mathrm{d}\theta.$$

a) Verify that the functions $\{f_n\}_n$ are orthonormal. Conclude that they are a basis for V and that V has dimension $2N+1$.

Solution

We will verify that pairwise inner products are 0 unless the pair is two of the same vector.

$$\langle f_n, f_m\rangle = \frac{1}{2\pi}\int_0^{2\pi} f_n(\theta)\overline{f_m(\theta)}\,\mathrm{d}\theta \tag{41}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{nj\theta}\overline{e^{mj\theta}}\,\mathrm{d}\theta \tag{42}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{nj\theta}e^{-mj\theta}\,\mathrm{d}\theta \tag{43}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{(n-m)j\theta}\,\mathrm{d}\theta \tag{44}$$

If $n = m$ then the integral is $\int_0^{2\pi}\mathrm{d}\theta$, and $\langle f_n, f_m\rangle = 1$. Otherwise we will use the FTC, observing that an antiderivative of $e^{(n-m)j\theta}$ is $\frac{1}{(n-m)j}e^{(n-m)j\theta}$.

$$= \frac{1}{2\pi}\left(\frac{1}{(n-m)j}\right) e^{(n-m)j\theta}\Big|_{\theta=0}^{\theta=2\pi} = 0 \tag{45}$$

As they are orthonormal, they are linearly independent. Because they are linearly independent and span V (by definition), they are a basis for V, and their number is the dimension of V.

b) Define the linear operator $D : V \to V$ by

$$Dv(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}v(\theta).$$

Verify that D is diagonal in the basis $\{f_n\}_n$. What are its eigenvalues?

Solution

We need to check that D acts on the basis vectors by scaling.

$$Df_n(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta}e^{jn\theta} \tag{46}$$

$$= (jn)\,e^{jn\theta} \tag{47}$$

$$= (jn)\,f_n(\theta) \tag{48}$$

The eigenvalue at eigenvector f_n is jn.

c) Does the fact that D is diagonal in an orthonormal basis imply that it is self-adjoint?

Solution

No. D must be diagonal with real eigenvalues, not merely diagonal.

d) Define the basis $\{g_n\}_n$ by

$$g_n = \sigma(n) j f_n = \begin{cases} j f_n, & n \geq 0 \\ -j f_n, & n < 0 \end{cases},$$

where $\sigma(n) = 1$ if n is nonnegative and -1 otherwise. Take $\{f_n\}_n$ as a basis for the domain of D and $\{g_n\}_n$ as a basis for the codomain of D. Verify that this choice of bases is a singular value decomposition of D. What are the singular values?

Solution

First we need to verify that $\{g_n\}_n$ is orthonormal.

$$\langle g_n, g_m \rangle = \langle \sigma(n) j f_n, \sigma(m) j f_m \rangle \tag{49}$$
$$= \sigma(n)\sigma(m) \left(j\right) \left(\overline{j}\right) \langle f_n, f_m \rangle \tag{50}$$
$$= \langle f_n, f_m \rangle \tag{51}$$

Because $\{f_n\}_n$ is orthonormal, so is $\{g_n\}_n$.

Next we need to verify that $D f_n = \alpha g_n$, where α is a nonnegative real number.

$$D f_n = \left(jn\right) f_n \tag{52}$$
$$= |n|\, g_n \tag{53}$$

These α_n also in non-increasing order, so they are the singular values.

7 Exam Proctoring Practice

Midterm 1 is on Friday, June 17th, and this will be the last homework you turn in before the exam. We are asking that as a part of this homework, you practice creating a proctoring recording, as you will be required to do for the exam. The requirements and instructions for the recording are the following. Pay careful attention to the order of the steps.

1 Use zoom to create your recording.

- Create a new meeting and make sure you are unmuted and video is on.
- Your video must capture as much of your face and work area as possible, within reason.
- You are permitted to use a virtual background if you would like.

2 Begin the recording.

3 Share screen.

- Make sure that your front-facing camera is still shown in the corner of your screen while screen sharing.

4 Open the exam and begin.

- For the purposes of this homework, move on to next step.

5 Submit your exam to gradescope.

- Submit your supplementary python file or ipython notebook for the exam to gradescope.
- For the purposes of this homework, submit this homework assignment and a "hello world" python/ipython notebook file to the HW3 and HW3 Supplementary Code assignments, respectively. (Therefore save this problem until you are ready to turn in your homework.)

6 Stop recording and upload.

- End the zoom meeting (so that video saves).
- (Optional) Upload the recording mp4 file to your Berkeley google drive.
- (Optional) Set sharing settings to share with the account `eecs16b-su20@berkeley.edu`.
- Copy the link of your recording, and submit the link to the following google form: ***Link to form***.
- This file should be submitted within 3 hours after completing the exam, since it may take some time to process the recording. If your file will not upload within this time limit, take a screenshot and send to an email to eecs16b-su20@berkeley.edu explaining the situation.
- If 3 hours after the end of the exam is later than 11pm in your timezone, send an email to eecs16b-su20@berkeley.edu explaining the situation, and submit the file by 9am the next morning in your timezone.

Important: we are asking you to do this exercise so you are familiar with the general procedure for audio/video proctoring before the exam happens. We reserve the right to modify this process if necessary before the actual exam.

If you are unable to perform or have concerns about any part of this procedure, please send an email to eecs16b-su20@berkeley.edu so we can attempt to resolve the issue before the exam.

8 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

d) **Do you have any feedback on this homework assignment?**

e) **Roughly how many total hours did you work on this homework?**