This homework is due on Tuesday, July 7, 2020, at 11:59PM.
Self-grades are due on Tuesday, July 14, 2020, at 11:59PM.

1 Complex Numbers

A common way to visualize complex numbers is to use the complex plane. Recall that a complex number z is often represented in Cartesian form.

$$z = x + jy \text{ with } \mathrm{Re}\{z\} = x \text{ and } \mathrm{Im}\{z\} = y$$

See Figure 1 for a visualization of z in the complex plane.

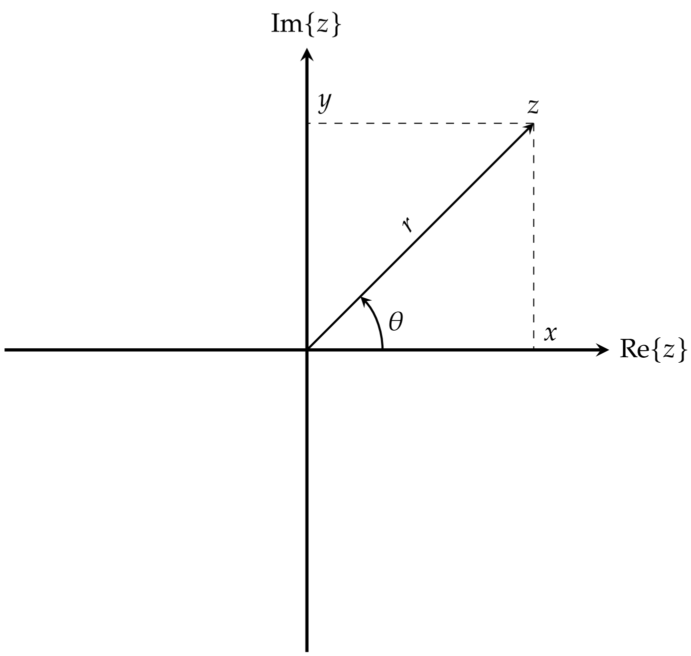


Figure 1: Complex Plane

In this question, we will derive the polar form of a complex number and use this form to make some interesting conclusions.

a) **Calculate the length of z in terms of x and y as shown in Figure 1.** This is the magnitude of a complex number and is denoted by $|z|$ or r. *Hint.* Use the Pythagorean theorem.

Solution

$$r = \sqrt{x^2 + y^2} = |z|$$

b) **Represent x, the real part of z, and y, the imaginary part of z, in terms of r and θ.**

Solution

$$x = r\cos(\theta) \text{ and } y = r\sin(\theta)$$

c) **Substitute for x and y in z.** Use Euler's identity[1] $e^{j\theta} = \cos\theta + j\sin\theta$ to **conclude that,**

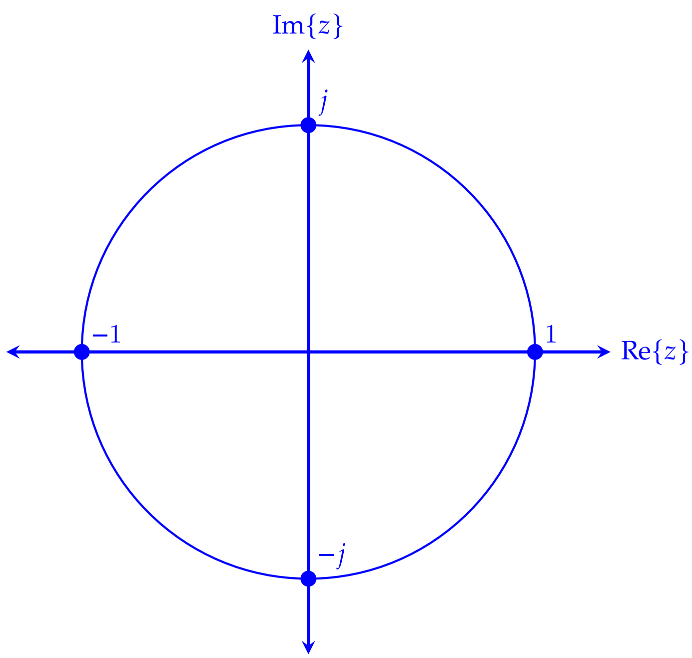
$$z = re^{j\theta}$$

Solution

$$\begin{aligned} z &= r\cos(\theta) + jr\sin(\theta) \\ &= r(\cos(\theta) + j\sin(\theta)) \\ &= re^{j\theta} \end{aligned}$$

d) In the complex plane, **draw out all the complex numbers such that $|z| = 1$. What are the z values where the figure intersects the real axis and the imaginary axis?**

Solution



e) If $z = re^{j\theta}$, **prove that $\bar{z} = re^{-j\theta}$.** Recall that the complex conjugate of a complex number $z = x + jy$ is $\bar{z} = x - jy$.

Solution

$$\begin{aligned} \bar{z} &= \overline{(r(\cos(\theta) + j\sin(\theta)))} \\ &= r(\cos(\theta) - j\sin(\theta)) \\ &= r(\cos(-\theta) + j\sin(-\theta)) \\ &= re^{-j\theta} \end{aligned}$$

[1] also known as de Moivre's Theorem.

f) **Show that,**

$$r^2 = z\bar{z}$$

Solution

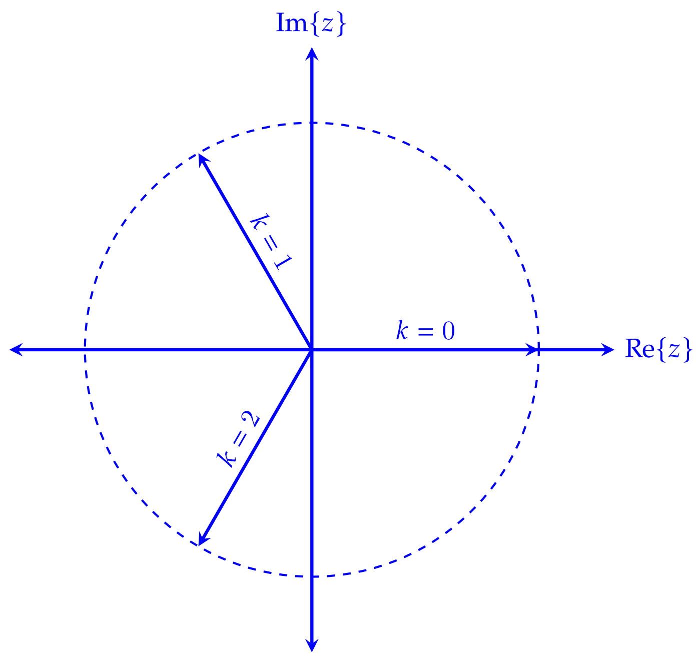
$$z\bar{z} = re^{j\theta}re^{-j\theta} = r^2e^{j\theta-j\theta} = r^2e^0 = r^2$$

g) Intuitively **argue that**

$$\sum_{k=0}^{2} e^{j\frac{2\pi}{3}k} = 0$$

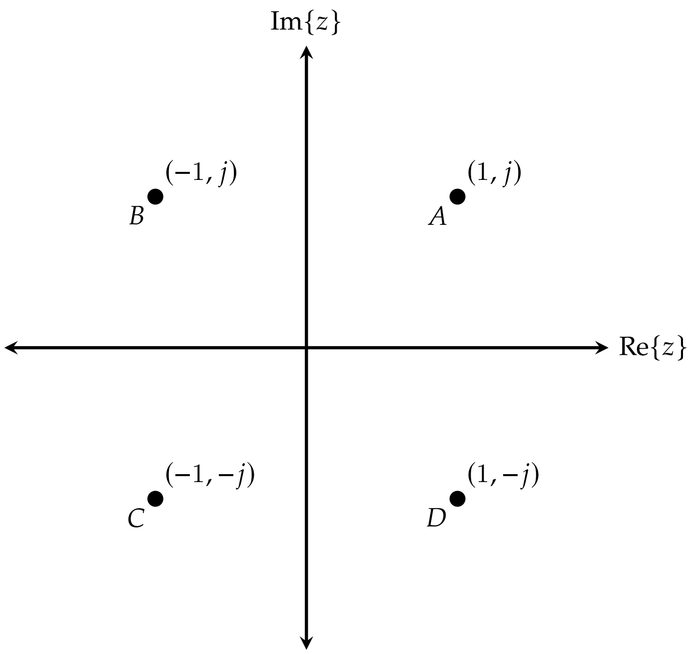
Do so by drawing out the different values of the sum in the complex plane and making an argument based on the vector sum.

Solution



The three vectors are of the same magnitude and are equally spaced from each other in a way that sums to zero. Intuitively, this is because we have three directions pointing perfectly away from each other. This can also be verified with Euler's formula (if you did that, give yourself full credit), but an intuitive argument based on the direction of the vectors is sufficient.

h) In modern wireless communication, signals are sent as complex exponentials $e^{j\omega t}$, with receivers detecting both the cosine and sine components of the signal. One common scheme for encoding digital data, such as what is used in your phone and in WiFi, is known as quadrature phase-shift keying (QPSK). In this technique, there are four points of interest on the complex plane (see figure below):



Write each of the points A, B, C, and D in polar form.

Solution

In Cartesian form, the complex numbers are given by:

$$A = 1 + j \qquad C = -1 - j$$
$$B = -1 + j \qquad D = 1 - j$$

For a given complex number $z = x + jy$, we employ the following two formulas to convert to polar form:

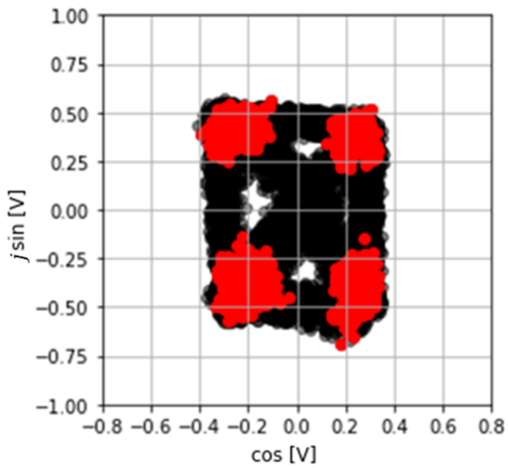
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \text{atan2}(y, x)$$

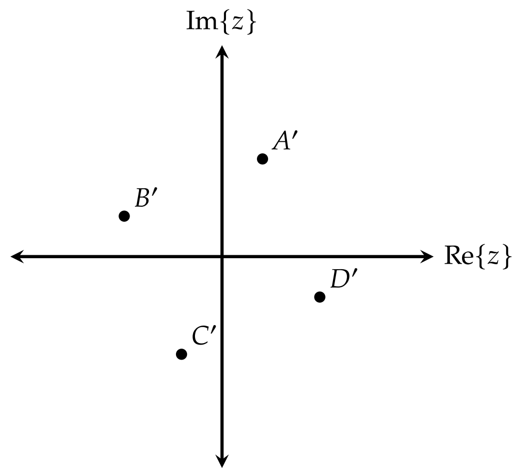
Plugging in, the final answer is:

$$A = \sqrt{2}e^{j\frac{\pi}{4}} \qquad C = \sqrt{2}e^{j\frac{-3\pi}{4}}$$
$$B = \sqrt{2}e^{j\frac{3\pi}{4}} \qquad D = \sqrt{2}e^{j\frac{-\pi}{4}}$$

i) Due to amplitude error and phase noise, in practice these points often arrive at the receiver scaled and rotated (see the figure on the left). Suppose the received points are as follows:

$$A' = 1e^{j\frac{3\pi}{8}} \qquad C' = 1e^{j\frac{-5\pi}{8}}$$
$$B' = 1e^{j\frac{7\pi}{8}} \qquad D' = 1e^{j\frac{-\pi}{8}}$$





Find a corrective value $r_x e^{j\theta_x}$ that when multiplied with A', B', C', D' recovers the original points A, B, C, D. What is the original noise $r_n e^{j\theta_n}$ that created the shift?

Solution

All the points are scaled and rotated by the same amount, so we pick A' arbitrarily. The problem boils down to finding $r_x e^{j\theta_x}$ such that

$$r_x e^{j\theta_x} A' = A$$

Plugging in the polar form of A and A',

$$r_x e^{j\theta_x} 1 e^{j\frac{3\pi}{8}} = r_x e^{j(\theta_x + \frac{3\pi}{8})} = \sqrt{2} e^{j\frac{\pi}{4}}$$

From which we can read off the solution,

$$r_x = \sqrt{2}$$
$$\theta_x = -\frac{\pi}{8}$$

This corrective value cancels out the rotation and scaling caused by the original noise, and thus the error is given by the multiplicative inverse:

$$r_n e^{j\theta_n} = \frac{1}{\sqrt{2}} e^{j\frac{\pi}{8}}$$

Note that the phase correction $e^{j\theta_x}$ is the complex conjugate of the phase noise $e^{j\theta_n}$ that created the error.

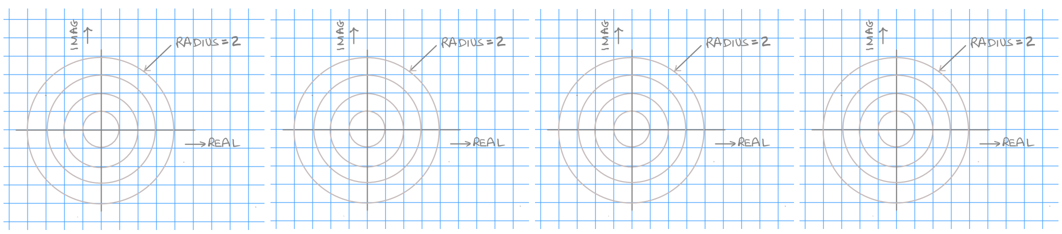
2 Phasors

a) Consider a resistor ($R = 1.5\Omega$), a capacitor ($C = 1\text{F}$), and an inductor ($L = 1\text{H}$) connected in series. **Give expressions for the impedances of Z_R, Z_C, Z_L for each of these elements as a function of the angular frequency ω.**

Solution

The impedances are as follows: $Z_R = R = 1.5$, $Z_C = \frac{1}{j\omega C} = \frac{1}{j\omega} = -\frac{j}{\omega}$ and $L = j\omega L = j\omega$.

b) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = \frac{1}{2}$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Then give the magnitude and phase of Z_{total}.**

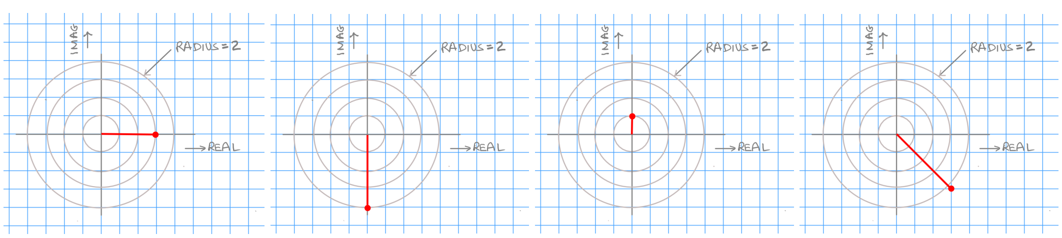


(a) $Z_R(@\omega = 0.5)$ (b) $Z_C(@\omega = 0.5)$ (c) $Z_L(@\omega = 0.5)$ (d) $Z_{total}(@\omega = 0.5)$

Figure 2: Impedances at $\omega = 0.5$.

Solution

Substituting for $\omega = \frac{1}{2}$ in the above answers, we get, $Z_R = 1.5$, $Z_C = -2j$ and $Z_L = 0.5j$. Since the elements are in series, $Z_{total} = Z_L + Z_C + Z_R = 1.5 - 1.5j$. This has magnitude $1.5\sqrt{2}$ and phase $-\frac{\pi}{4}$. Following are the plots:



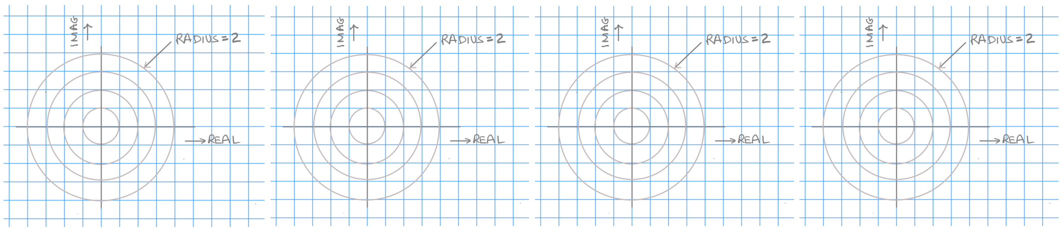
(a) $Z_R(@\omega = 0.5)$ (b) $Z_C(@\omega = 0.5)$ (c) $Z_L(@\omega = 0.5)$ (d) $Z_{total}(@\omega = 0.5)$

Figure 3: Impedances at $\omega = 0.5$.

c) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = 1$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Then give the magnitude and phase of Z_{total}.**

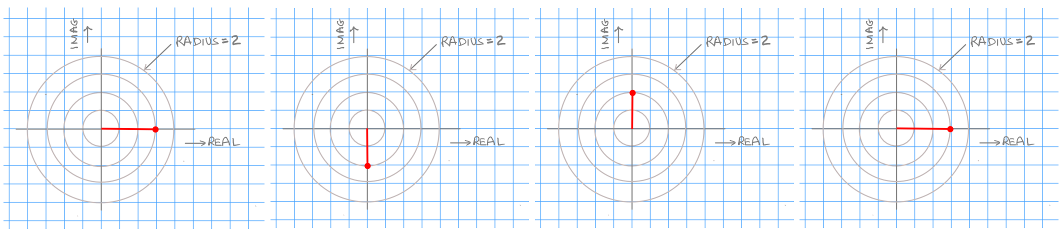
Solution

Following the same method as last time, with $\omega = 1$, $Z_R = 1.5$, $Z_C = -j$, $Z_L = j$ and $Z_{total} = 1.5$ Thishas magnitude 1.5 and phase 0.



(a) $Z_R(@\omega = 1)$ (b) $Z_C(@\omega = 1)$ (c) $Z_L(@\omega = 1)$ (d) $Z_{total}(@\omega = 1)$

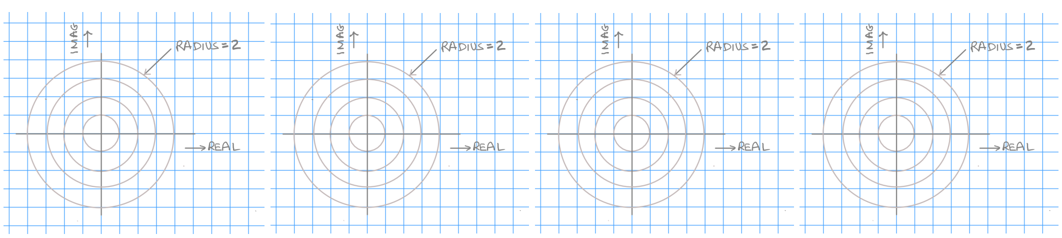
Figure 4: Impedances at $\omega = 1$.



(a) $Z_R(@\omega = 1)$ (b) $Z_C(@\omega = 1)$ (c) $Z_L(@\omega = 1)$ (d) $Z_{total}(@\omega = 1)$

Figure 5: Impedances at $\omega = 1$.

d) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = 2$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Then give the magnitude and phase of Z_{total}.**

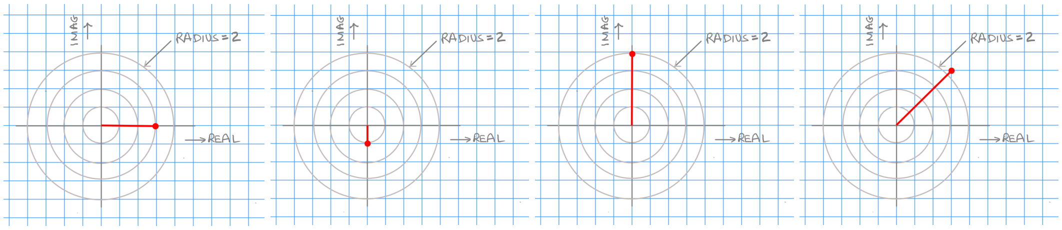


(a) $Z_R(@\omega = 2)$ (b) $Z_C(@\omega = 2)$ (c) $Z_L(@\omega = 2)$ (d) $Z_{total}(@\omega = 2)$

Figure 6: Impedances at $\omega = 2$.

Solution

Again, following the same method as last time, with $\omega = 2$, $Z_R = 1.5$, $Z_C = -0.5j$, $Z_L = 2j$ and $Z_{total} = 1.5 + 1.5j$. This has magnitude $1.5\sqrt{2}$ and phase $+\frac{\pi}{4}$.



(a) $Z_R(@\omega = 2)$ (b) $Z_C(@\omega = 2)$ (c) $Z_L(@\omega = 2)$ (d) $Z_{total}(@\omega = 2)$

Figure 7: Impedances at $\omega = 2$.

e) **For the previous series combination of RLC elements, what is the "natural frequency" ω_n where the series impedance is purely real?**

Solution

From our above answers, clearly the natural frequency, $\omega_n = 1$ rad/s . This is where the imaginary parts of the impedance cancel each other.

3 Color Organ Filter Design

In the third lab, we will design low-pass and high-pass filters for a color organ. There are red, and green LEDs. Each color will correspond to a specified frequency range of the input audio signal. The intensity of the light emitted will correspond to the amplitude of the audio signal for that frequency range.

a) First, you realize that you can build simple filters using a resistor and a capacitor. Design the first-order **passive** low and high pass filters with following frequency ranges for each filter using $1\,\mu\mathrm{F}$ capacitors. ("Passive" means that the circuit components making up the filter do not require any power supply.)

- Low pass filter – 3-dB frequency at $2400\,\mathrm{Hz} = 2\pi \cdot 2400\frac{rad}{sec}$
- High pass filter – 3-dB frequency at $100\,\mathrm{Hz} = 2\pi \cdot 100\frac{rad}{sec}$

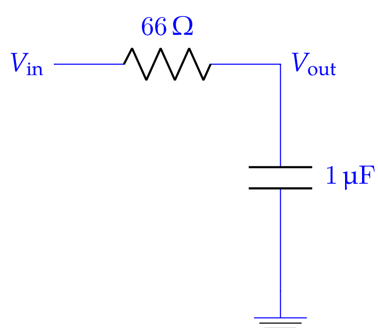
Draw the schematic-level representation of your designs and show your work finding the resistor values. Also, please mark V_{in}, V_{out}, and ground nodes in your schematic. Round your results to two significant figures. Remember that $\omega = 2\pi f$!

Solution

a) Low-pass filter

$$f_{3\,\mathrm{dB}} = \frac{1}{2\pi RC} = 2400\,\mathrm{Hz}$$

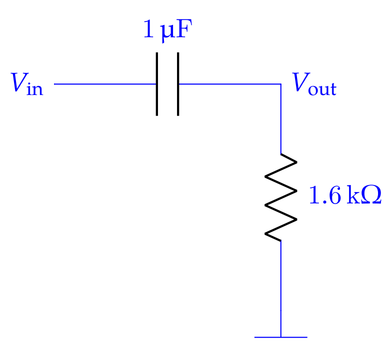
Therefore, we need a $66\,\Omega$ resistor.



b) High-pass filter

$$f_{3\,\mathrm{dB}} = \frac{1}{2\pi RC} = 100\,\mathrm{Hz}$$

Therefore, we need a $1.6\,\mathrm{k}\Omega$ resistor.



b) Bode plots are a useful tool to understand the frequency response of any filter you might design. First, write out the transfer functions $H_{\text{LPF}} = \frac{V_{\text{out, LPF}}}{V_{\text{in, LPF}}}$ and $H_{\text{HPF}} = \frac{V_{\text{out, HPF}}}{V_{\text{in, HPF}}}$ for both filters designed in part (a). Then, use a computer to draw the Bode **magnitude** plot from 0.1 Hz to 1 GHz.

If you use Python, plot 10,000 samples of the magnitude of the bode plot. Here are a couple useful functions (*make sure to lookup documentation for each of these!*):

- `scipy.signal.TransferFunction`
- `scipy.signal.bode`
- `np.linspace`
- `matplotlib.pyplot`

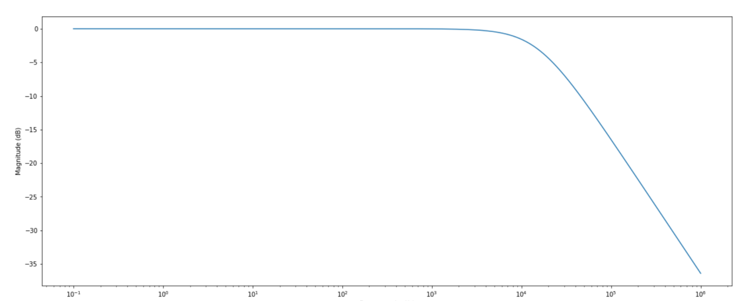
Solution

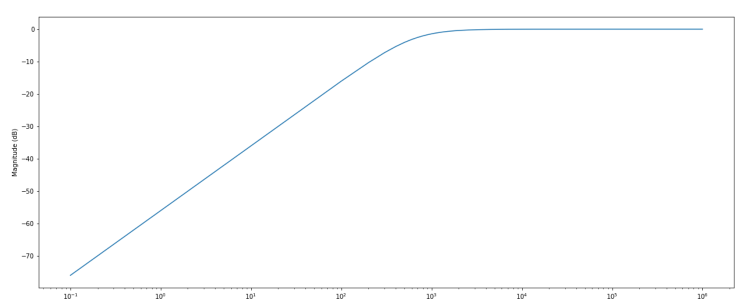
From a phasor domain perspective, both the high pass and low pass filters are voltage dividers. Let $Z_C = \frac{1}{j\omega C}$ and $Z_R = R$ be the impedances of a capacitor and resistor respsectively. Then,

$$V_{\text{out, LPF}} = V_{\text{in, LPF}} \frac{Z_C}{Z_R + Z_C} \rightarrow H_{\text{LPF}} = \frac{Z_C}{Z_R + Z_C} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{j\omega RC + 1}$$

$$V_{\text{out, HPF}} = V_{\text{in, HPF}} \frac{Z_R}{Z_R + Z_C} \rightarrow H_{\text{HPF}} = \frac{Z_R}{Z_R + Z_C} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{j\omega RC + 1}$$

Plugging in $R = 66\,\Omega$ for the low pass filter and $R = 1.6\,\text{k}\Omega$ for the high pass and we can draw the magnitudes of the bode plots as below.



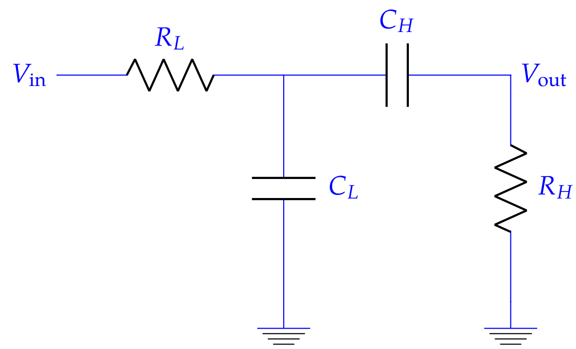


Notice that the -3dB point for each filter lines up with its cutoff frequency ($\omega_c = \frac{1}{RC}$)!

c) You decide to build a bandpass filter by simply cascading the first-order low-pass and high-pass filters you designed in part (a). Connect the V_{out} node of your low-pass filter directly to the V_{in} node of your high pass filter. The V_{in} of your new band-pass filter is the V_{in} of your old low-pass filter, and the V_{out} of the new filter is the V_{out} of your old high-pass filter.

What is H_{BPF}, the transfer function of your new band-pass filter? Use R_L, C_L, R_H, and C_H for low-pass filter and high-pass filter components, respectively. Show your work.

Solution



$$\left(\frac{1}{j\omega C_H} + R_H\right) \parallel \frac{1}{j\omega C_L} = \frac{\left(\frac{1}{j\omega C_H} + R_H\right)\frac{1}{j\omega C_L}}{\frac{1}{j\omega C_L} + \frac{1}{j\omega C_H} + R_H} = \frac{1 + j\omega R_H C_H}{-\omega^2 R_H C_L C_H + j\omega(C_H + C_L)}$$

Therefore, the transfer function from V_{in} of **the low pass filter** to V_{out} of **the low pass filter** is

$$H_{LPF} = \frac{\left(\frac{1}{j\omega C_H} + R_H\right) \parallel \frac{1}{j\omega C_L}}{R_L + \left(\frac{1}{j\omega C_H} + R_H\right) \parallel \frac{1}{j\omega C_L}} = \frac{1 + j\omega R_H C_H}{1 - \omega^2 R_L R_H C_L C_H + j\omega(R_H C_H + R_L C_L + R_L C_H)}$$

And, the transfer function from V_{out} of **the low pass filter** to V_{out} of **the high pass filter** is

$$H_{HPF} = \frac{j\omega R_H C_H}{1 + j\omega R_H C_H}$$

The overall transfer function is

$$H_{BPF} = H_{LPF} \cdot H_{HPF} = \frac{j\omega R_H C_H}{1 - \omega^2 R_L R_H C_L C_H + j\omega(R_H C_H + R_L C_L + R_L C_H)}$$

d) Plug the component values you found in (a) into the transfer function H_{BPF}. **Then, using a computer via Python or your favorite graphing tool, draw the Bode magnitude plot from** $0.1\,\text{Hz}$ **to** $1\,\text{GHz}$**.**

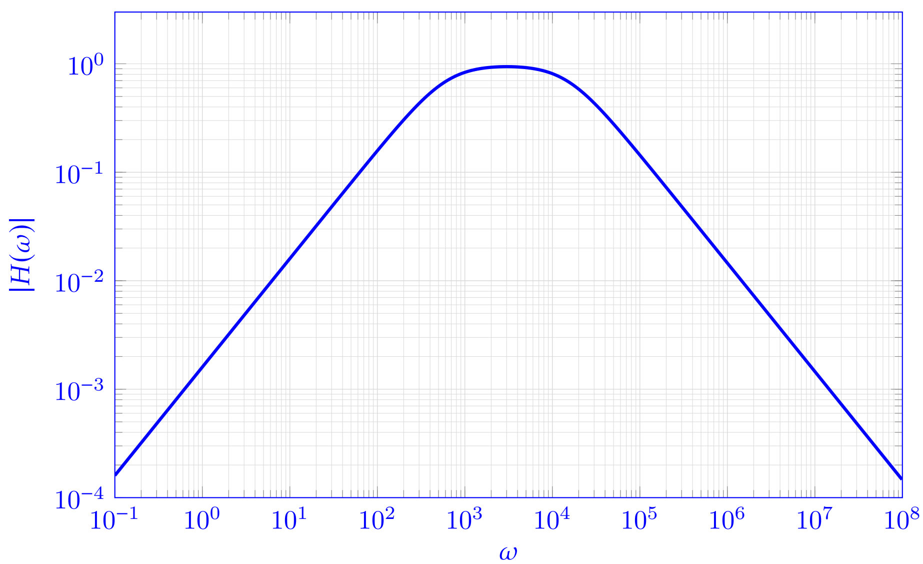
What is the maximum magnitude of H_{BPF}? Is that something that you want? If not, explain why not and suggest a simple way (either adding passive or active components) to fix it.

Solution

$$R_H C_H = 1.6\,\mathrm{k\Omega} \cdot 1\,\mathrm{\mu F} = 1.6 \cdot 10^{-3}\mathrm{s}$$
$$R_L C_L = 66\,\Omega \cdot 1\,\mathrm{\mu F} = 6.6 \cdot 10^{-5}\mathrm{s}$$
$$R_L C_H = 66\,\Omega \cdot 1\,\mathrm{\mu F} = 6.6 \cdot 10^{-5}\mathrm{s}$$

$$H_{BPF} = \frac{j\omega(1.6 \cdot 10^{-3})}{1 - \omega^2(1.1 \cdot 10^{-7}) + j\omega(1.7 \cdot 10^{-3})}$$

The Bode plot is as below.



There are two roots for denumerator and one root for numerator at 100 Hz, 2.4 kHz, and DC, respectively. The maximum magnitude (around $500\,\mathrm{Hz} = 3.14 \times 10^3\ \frac{\mathrm{rad}}{\mathrm{s}}$) is

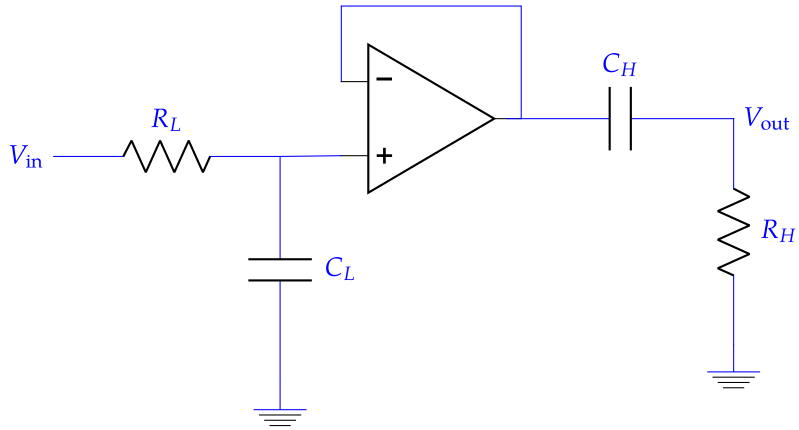
$$\left| \frac{j(3.14 \cdot 10^3)(1.6 \cdot 10^{-3})}{1 - (3.14 \cdot 10^3)^2(1.1 \cdot 10^{-7}) + j(3.14 \cdot 10^3)(1.7 \cdot 10^{-3})} \right| = 0.94\,\frac{\mathrm{V}}{\mathrm{V}}$$

This is pretty similar to what we wanted. The gain, $|H_{BPF}|$, is close to 1 at its maximum. However, the transfer function of the bandpass filter that we likely intended to get by cascading the two filter circuits was:

$$\begin{aligned} H_{\text{ideal BPF}} &= \frac{j\omega R_H C_H}{(1 + j\omega R_H C_H)(1 + j\omega R_L C_L)} \\ &= \frac{j\omega R_H C_H}{1 - \omega^2 R_H C_H R_L C_L + j\omega(R_L C_L + R_H C_H)} \end{aligned}$$

Therefore, in our circuit, only the $j\omega R_L C_H$ term is added at the denominator. Because $R_L = 66\,\Omega$ is small, it did not cause any significant problem in our case. $j\omega R_L C_H$ is added because the low pass filter is experiencing impedance loading from the high pass filter, leading to a change in H_{LPF}. However, to be safe, a simple solution is to place a voltage buffer between the filters as below.

Note that the ideal voltage buffer has infinite input impedance and zero output impedance. This blocks any load effects from the following stage, and the next stage will see the op-amp output as an ideal voltage source.


C_H
V_{out}
R_L
V_{in}
R_H
C_L

4 Transfer Functions and Filters

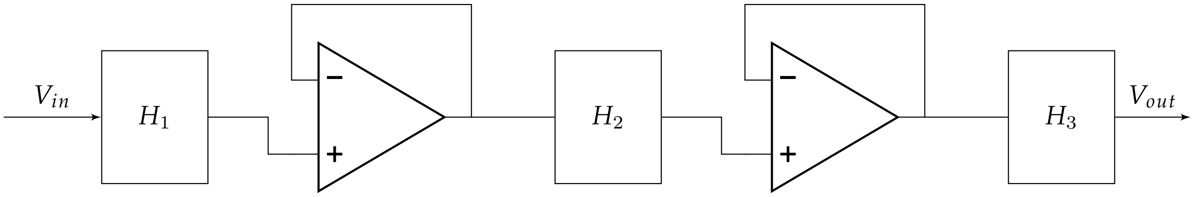


Figure 8: Three filters cascaded via unity-gain op-amp buffers

Suppose that at some frequency ω_0 radians/sec we know that:

$$H_1(j\omega_0) = 3e^{j\frac{\pi}{4}} \quad H_2(j\omega_0) = \frac{1}{2}e^{-j\frac{\pi}{3}} \quad H_3(j\omega_0) = 4e^{j\frac{5\pi}{6}}$$

If $V_{in}(t) = 2\sin\left(\omega_0 t + \frac{\pi}{2}\right)$:

a) What is the phasor $\widetilde{V}_{in}$ of the input voltage?

b) What is the phasor $\widetilde{V}_{out}$ of the output voltage?

c) What is $V_{out}(t)$?

Solution

First we find the phasor representation of $V_{in}(t)$. Remember that $sin(t + \frac{\pi}{2}) = cos(t)$:

$$V_{in}(t) = 2cos(\omega_0 t)$$

$$\widetilde{V}_{in} = 2e^{j0} = 2$$

Then to get the overall transfer function, multply all the individual transfer functions. This is the same as multplying the phasor magnitudes and adding the phases:

$$H_{total} = \frac{V_{out}}{V_{in}} = H_1 \cdot H_2 \cdot H_3 = 6e^{j\frac{3\pi}{4}}$$

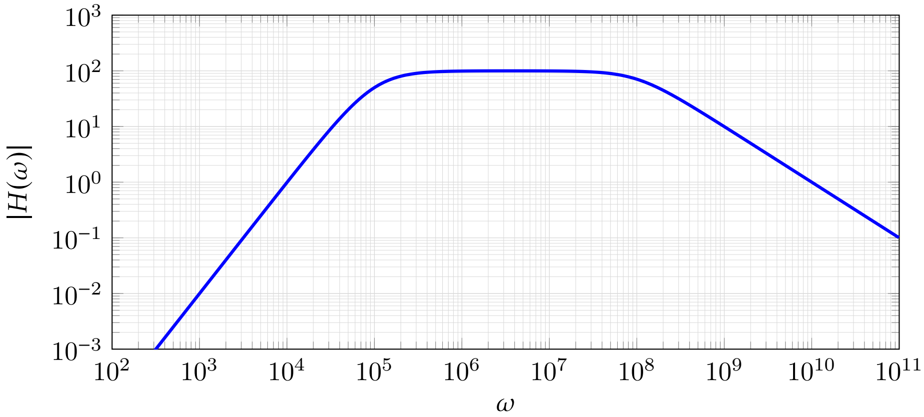
From phasor analysis, $\widetilde{V}_{out} = H_{total}\widetilde{V}_{in} = 12e^{j\frac{3\pi}{4}}$

Finally we convert back to the original time domain:

$$V_{out}(t) = 12cos(\omega_0 t + \frac{3\pi}{4})$$

The phase shift $\frac{3\pi}{4}$ is the phase of the output voltage phasor. We know this from the equation for cosine: $\cos\theta = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$.

d) The Bode magnitude plot of the $H(\omega)$ is the following.



What is $H(\omega)$?

Solution

The cutoff frequencies are at $\omega_{c1} = 10^5$ and $\omega_{c2} = 10^8$. The magnitude increases by a factor of 100 per decade and then flattens out at $\omega_{c1} = 10^5$. Then at $\omega_{c2} = 10^8$ we see a dropoff of 10 per decade.

This means we have a double zero at the origin, a double pole at $\omega_{c1} = 10^5$ and then a single pole at $\omega_{c2} = 10^8$. As a result the transfer function should be

$$H(\omega) = K \cdot (j\omega)^2 \cdot \frac{1}{(1 + j\omega/10^5)^2} \cdot \frac{1}{(1 + j\omega/10^8)}$$

To determine the value of K we know that the transfer function $H(\omega)$ at $\omega = 10^4$ has magnitude equal to 1.

$$\left|H(\omega = 10^4)\right| = K \cdot (10^4)^2 \cdot \frac{1}{1^2 + (0.1)^2} \cdot \frac{1}{\sqrt{1 + (10^{-4})^2}} = 1$$

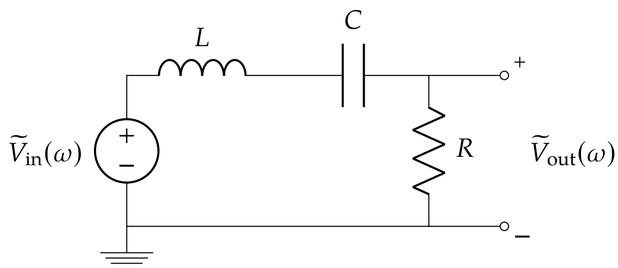
We will approximate $(0.1)^2$ and 10^{-8} as zero and this implies that $K = 10^{-8}$.

We conclude by saying the transfer function $H(\omega)$ is

$$H(\omega) = 100 \cdot \frac{(j\omega/10^5)^2}{(1 + j\omega/10^5)^2} \cdot \frac{1}{1 + j\omega/10^8}$$

5 Band-pass filter

It is quite common to need to design a filter which selects only a narrow range of frequencies. One example is in WiFi radios, it is desirable to select only the 2.4GHz frequency containing your data, and reject information from other nearby cellular or bluetooth frequencies. This type of filter is called a band-pass filter; we will explore the design of this type of filter in this problem.



a) **Write down the impedance of the series RLC combination in the form** $Z_{RLC}(\omega) = A(\omega) + jX(\omega)$**, where** $X(\omega)$ **is a real valued function of** ω**.**

Solution

Since the capacitor, resistor and inductor are in series, the equivalent impedance is given by,

$$Z_{RLC}(\omega) = R + Z_L(\omega) + Z_C(\omega)$$
$$\implies Z_{RLC}(\omega) = R + j\omega L + \frac{1}{j\omega C}$$

Since,

$$\frac{1}{j} = -j$$
$$Z_{RLC}(\omega) = R + j(\omega L - \frac{1}{\omega C})$$

Hence,

$$A(\omega) = R$$

and

$$X(\omega) = \omega L - \frac{1}{\omega C}$$

b) **Write down the transfer function** $H(\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)}$ **for this circuit.**

Solution

Using the same voltage divider rule we've used in the past, V_{out} is:

$$\widetilde{V}_{out} = \frac{Z_R}{Z_{RLC}}\widetilde{V}_{in}$$
$$H(\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} = \frac{R}{Z_{RLC}(\omega)}$$
$$H(\omega) = \frac{R}{R + j(\omega L - \frac{1}{\omega C})}$$

c) **At what frequency ω_n does $X(\omega_n) = 0$?** (i.e. at what frequency is the impedance of the series combination of RLC purely real — meaning that the imaginary terms coming from the capacitor and inductor completely cancel each other.)

What happens to the relative magnitude of the impedances of the capacitor and inductor as ω moves above and below ω_n? What is the value of the transfer function at this frequency?

Solution

$$X(\omega_n) = 0 = \omega_n L - \frac{1}{\omega_n C}$$

Multiplying both sides by ω_n:

$$0 = \omega_n^2 L - \frac{1}{C}$$

$$\omega_n = \frac{1}{\sqrt{LC}}$$

As the frequency increases above ω_n, the magnitude of the impedance of the inductor gets higher while the capacitor gets lower. Since the two components are in series, the impedance of the inductor will dominate. As the frequency decreases below ω_n, the magnitude of the impedance of the capacitor gets higher while the inductor gets lower. In this case the impedance of the capacitor will dominate.

At ω_n, $Z_{RLC} = R$, since the imaginary components cancel out perfectly. As a result

$$H(\omega_n) = 1$$

d) In most filters, we are interested in the cutoff frequency, since that helps define the frequency range over which the filter operates. Remember that this is the frequency at which the magnitude of the transfer function drops by a factor of $\sqrt{2}$ from its maximum value. Notice that the real part of the impedance Z_{RLC} is not changing with frequency and stays at R. What we care about is the frequencies where the imaginary part of the impedance equals $-jR$ to $+jR$.

Solve for the cutoff frequencies by finding ω_{c1} and ω_{c2} where the imaginary part of the impedance $Z_{RLC} = \pm R$. Then use these cutoff frequencies to calculate the bandwidth $\Delta\omega = |\omega_{c1} - \omega_{c2}|$.

Solution

From the first part we know that,

$$X(\omega) = \omega L - \frac{1}{\omega C}$$

We are looking for values of ω where $X(\omega) = \pm R$ or

$$\omega L - \frac{1}{\omega C} = \pm R$$

Multiplying both sides by ωC we get

$$\omega^2 LC - 1 = \pm \omega RC$$

We will break this into two cases where

$$\omega^2 LC - 1 = \omega RC$$
$$\omega^2 LC - 1 = -\omega RC$$

This will give two quadratic equations of the form

$$\omega^2 - \frac{R}{L}\omega - \frac{1}{LC} = 0$$
$$\omega^2 + \frac{R}{L}\omega - \frac{1}{LC} = 0$$

Using the quadratic formula, we end up with four solutions

$$\omega = \frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}$$
$$\omega = -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}}$$

Discarding the negative solutions, we see that the cutoff frequencies are

$$\omega = \frac{1}{2}\sqrt{\left(\frac{R}{L}\right)^2 + \frac{4}{LC}} \pm \frac{R}{2L}$$

Using these cutoff frequencies, it follows that the bandwidth is

$$\Delta\omega = |\omega_{c1} - \omega_{c2}| = \frac{R}{L}$$

e) **Simplify $X(\omega)$ in two cases, when $\omega \to \infty$ and when $\omega \to 0$. Plug this simplified $X(\omega)$ into your previously solved expressions to find the transfer function at high and low frequencies.**

Solution

At low frequencies,

$$X(\omega) \approx -\frac{1}{\omega C}$$

Plugging this back in to the original transfer function we get a CR high-pass filter.

$$H(\omega) \approx \frac{R}{R - j\frac{1}{\omega C}}$$

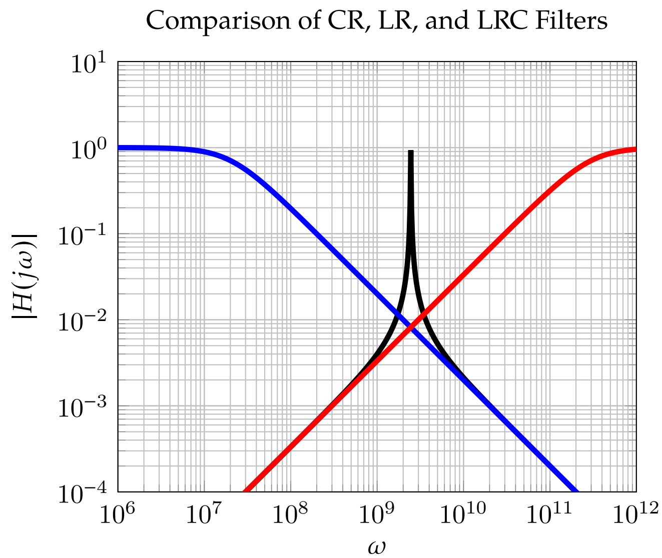
At high frequencies,

$$X(\omega) \approx \omega L$$

Plugging this back in to the original transfer function we get a LR low-pass filter.

$$H(\omega) \approx \frac{R}{R + j\omega L}$$

f) **A bode plot for a CR filter, a LR filter, and a LCR filter is shown for $L = 9nH$, $R = 0.18\Omega$, and $C = 18.7pF$. Assign each filter to its corresponding line on the plot. Label the locations of the corner frequencies, and describe the behavior of the LCR filter at very high and low frequencies.**



Solution

The CR high pass filter is the red line, the LR blue, and the LCR black. Our band-pass filter looks like an LR low-pass filter at high frequencies and a CR high-pass filter at low frequencies. Note that in this case, the cutoff frequencies for the LR and CR filters are not the same as LCR cutoff frequencies or the resonance frequency. In the vicinity of the resonance frequency, notice that the slope of the filter is much larger than any first-order LR or RC filter. This allows for better rejection of frequencies outside of the desired frequency.

The corner frequency for the LR filter is $\frac{R}{L} = 20\text{Mrad/s}$ and for the CR filter is $\frac{1}{RC} = 300\text{Grad/sec}$. The two frequencies at which the magnitude of the LRC transfer function equals $\frac{1}{\sqrt{2}}$ are $2.4\text{Grad/s} \pm 10\text{Mrad/s}$.

6 Matrix Differential Equations

In this problem, we consider ordinary differential equations which can be written in the following form

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \tag{1}$$

where x, y are variables depending on t, $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$, and A is a 2×2 matrix with constant coefficients. We call (1) a matrix differential equation.

a) Suppose we have a system of ordinary differential equations

$$x' = 7x - 8y \tag{2}$$
$$y' = 4x - 5y \tag{3}$$

Write this in the form of (1).

Solution

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 7 & -8 \\ 4 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \tag{4}$$

b) Compute the eigenvalues of the matrix A from the previous part.

Solution

The characteristic polynomial of A is

$$\begin{aligned} \det\left(\begin{bmatrix} 7-\lambda & -8 \\ 4 & -5-\lambda \end{bmatrix}\right) &= (7-\lambda)(-5-\lambda) + 32 \\ &= \lambda^2 - 7\lambda + 5\lambda - 35 + 32 \\ &= \lambda^2 - 2\lambda - 3 \\ &= (\lambda + 1)(\lambda - 3). \end{aligned}$$

Thus the eigenvalues of A are $\lambda = -1, 3$.

c) We claim that the solution for $x(t), y(t)$ is of the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_0 t} + c_3 e^{\lambda_1 t} \end{bmatrix},$$

where c_0, c_1, c_2, c_3 are constants, and λ_0, λ_1 are the eigenvalues of A. Suppose that the initial conditions are $x(0) = 1$, $y(0) = -1$. Solve for the constants c_0, c_1, c_2, c_3.

Hint: What are $\frac{dx}{dt}(0)$ *and* $\frac{dy}{dt}(0)$?

Solution

Substituting the eigenvalues computed in the previous part, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_0 e^{-t} + c_1 e^{3t} \\ c_2 e^{-t} + c_3 e^{3t} \end{bmatrix}.$$

At $t = 0$, we have $x(0) = 1, y(0) = -1$ so we can compute its derivatives' initial conditions as

$$\begin{bmatrix} x'(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 7x(0) - 8y(0) \\ 4x(0) - 5y(0) \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \end{bmatrix}.$$

$$1 = c_0 + c_1 \tag{5}$$
$$15 = -c_0 + 3c_1 \tag{6}$$
$$-1 = c_2 + c_3 \tag{7}$$
$$9 = -c_2 + 3c_3 \tag{8}$$

This gives $c_0 = -3$, $c_1 = 4$, $c_2 = -3$, and $c_3 = 2$. Thus we have

$$x(t) = -3e^{-t} + 4e^{3t} \tag{9}$$
$$y(t) = -3e^{-t} + 2e^{3t} \tag{10}$$

d) Verify that the solution for $x(t), y(t)$ found in the previous part satisfies the original system of differential equations (2), (3).

Solution

We compute the derivative of x with respect to t in (9) to get

$$x'(t) = 3e^{-t} + 12e^{3t}.$$

The right hand side of (2) is

$$7x - 8y = -21e^{-t} + 28e^{3t} + 24e^{-t} - 16e^{3t} = 3e^{-t} + 12e^{3t}$$

hence our solution for $x(t)$ satisfies (2).

Similarly, we compute the derivative of y with respect to t in (10) to get

$$y'(t) = 3e^{-t} + 6e^{3t}.$$

The right hand side of (3) is

$$4x - 5y = -12e^{-t} + 16e^{3t} + 15e^{-t} - 10e^{3t} = 3e^{-t} + 6e^{3t}$$

hence our solution for $y(t)$ satisfies (3).

e) We now apply the method above to solve another second-order ordinary differential equation. Suppose we have the system

$$z''(t) - 5z'(t) + 6z(t) = 0, \tag{11}$$

where $z' = \frac{dz}{dt}$ and $z'' = \frac{d^2z}{dt^2}$. Write this in the form of (1), by choosing your state variables to be $x(t) = z(t)$, $y(t) = z'(t)$.

Solution

If we set $x(t) = z(t)$, $y(t) = z'(t)$, then we have

$$x'(t) = z'(t) = y(t) \tag{12}$$
$$y'(t) = z''(t) = 5z'(t) - 6z(t) = 5y(t) - 6x(t) \tag{13}$$

We can write this in the form of (1) as follows

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}, \tag{14}$$

f) Solve the system in (11) with the initial conditions $z(0) = 1$, $z'(0) = 1$, using the method developed in parts (b) and (c).

Solution

We first compute the eigenvalues of the matrix from the previous part. The characteristic polynomial is

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{bmatrix}\right) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)\,.$$

Thus the eigenvalues are $\lambda = 2, 3$.

From part (c), the solution for $x(t)$, $y(t)$ is of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_0 e^{2t} + c_1 e^{3t} \\ c_2 e^{2t} + c_3 e^{3t} \end{bmatrix}.$$

At $t = 0$, we have $x(0) = z(0) = 1$ and $x'(0) = z'(0) = 1$:

$$1 = c_0 + c_1 \tag{15}$$
$$1 = 2c_0 + 3c_1 \tag{16}$$

This gives $c_0 = 2$ and $c_1 = -1$. Thus we have

$$x(t) = 2e^{2t} - e^{3t} \tag{17}$$
$$y(t) = x'(t) = 4e^{2t} - 3e^{3t} \tag{18}$$

Hence we have the solution

$$z(t) = 2e^{2t} - e^{3t}\,.$$

7 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **Watch any of the 16A review discussions (1A or 1B, found here), and explain one of the following topics in your own words: passive sign convention, nodal analysis, superposition, Thevenin and Norton equivalent circuits, or op-amps in negative feedback. More 16A circuits resources can be found at `https://www.eecs16a.org`, notes 11-20.**

b) **What sources (if any) did you use as you worked through the homework?**

c) **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

d) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

e) **Do you have any feedback on this homework assignment?**

f) **Roughly how many total hours did you work on this homework?**