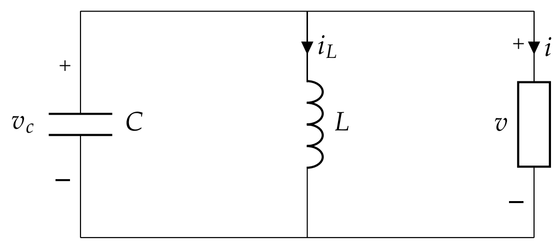
This homework is due on August 4, 2020, at 11:59PM.
Self-grades are due on Tuesday, August 11, 2020, at 11:59PM.

1 Nonlinear circuit component

Note: Solutions to this problem will be released on Wednesday, but you are still required to submit your own original work for this problem.

Consider the circuit below that consists of a capacitor, inductor, and a third element with a nonlinear voltage-current characteristic:

$$i = 2v - v^2 + 4v^3$$



a) Write a state space model of the form

$$\frac{dx_1(t)}{dt} = f_1(x_1(t), x_2(t))$$

$$\frac{dx_2(t)}{dt} = f_2(x_1(t), x_2(t))$$

Where $x_1(t) = v_c(t)$ and $x_2(t) = i_L(t)$.

b) Linearize the state model at the equilibrium point $x_1 = x_2 = 0$ and specify the resulting A matrix.

c) Is the linearized system stable?

2 Discrete Time Control

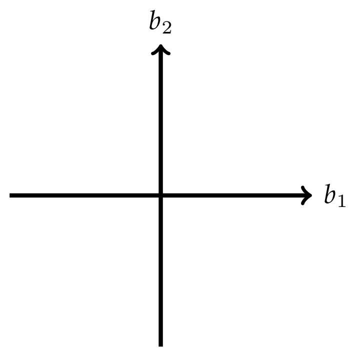
Note: Solutions to this problem will be released on Wednesday, but you are still required to submit your own original work for this problem.

Consider the system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} 1.5 & 1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)$$

a) Determine if the system is stable.

b) Determine the set of all (b_1, b_2) values for which the system is **not** controllable and sketch this set of points in the b_1-b_2 plane below.



3 Balance — linearizing a vector system

Note: Solutions to this problem will be released on Wednesday, but you are still required to submit your own original work for this problem.

Justin is working on a small jumping robot named Salto. Salto can bounce around on the ground, but Justin would like Salto to balance on its toe and stand still. In this problem, we'll work on systems that could help Salto balance on its toe using its reaction wheel tail.

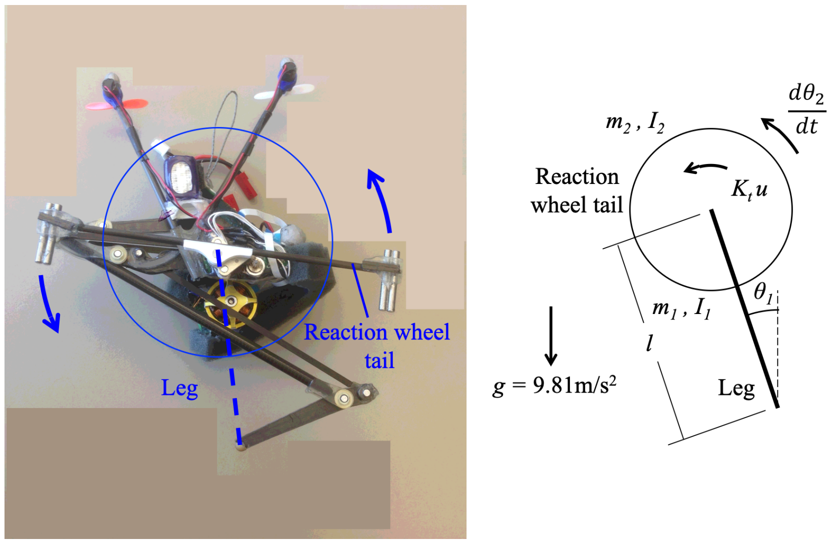


Figure 1: Picture of Salto and the x-z physics model. You can watch a video of Salto here: `https://youtu.be/ZFGxnF9SqDE`

Standing on the ground, Salto's dynamics in the x-z plane (called the sagittal plane in biology) look like an inverted pendulum with a flywheel on the end,

$$(I_1 + (m_1 + m_2)l^2)\frac{\mathrm{d}^2\theta_1(t)}{\mathrm{d}t^2} = -K_t u(t) + (m_1 + m_2)lg\sin\big(\theta_1(t)\big)$$
$$I_2\frac{\mathrm{d}^2\theta_2(t)}{\mathrm{d}t^2} = K_t u(t),$$

where $\theta_1(t)$ is the angle of the robot's body relative to the ground at time t ($\theta_1 = 0\,\mathrm{rad}$ means the body is exactly vertical), $\frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t}$ is the robot body's angular velocity, $\frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t}$ is the angular velocity of the reaction wheel tail, and $u(t)$ is the current input to the tail motor. $m_1, m_2, I_1, I_2, l, K_t$ are positive constants representing system parameters (masses and angular momentums of the body and tail, leg length, and motor torque constant, respectively) and $g = 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}$ is the acceleration due to gravity.

Numerically substituting Salto's physical parameters, the differential equations become:

$$0.001\frac{\mathrm{d}^2\theta_1(t)}{\mathrm{d}t^2} = -0.025u(t) + 0.1\sin\big(\theta_1(t)\big)$$
$$5(10^{-5})\frac{\mathrm{d}^2\theta_2(t)}{\mathrm{d}t^2} = 0.025u(t)$$

a) Using the state vector $\begin{bmatrix} \theta_1 \\ \frac{\mathrm{d}\theta_1(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t} \end{bmatrix}$, and input u, linearize the system about the point

$\vec{x}^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ with nominal input $u^* = 0$. **Write the linearized equation as $\frac{d}{dt}\vec{x} = A\vec{x} + Bu$. Write out the matrices with the physical numerical values.**

Note: Since the tail is like a wheel, we care only about the tail's angular velocity $\frac{\mathrm{d}\theta_2(t)}{\mathrm{d}t}$ and not its angle $\theta_2(t)$. This is why $\theta_2(t)$ is not a state.

Hint: The $\sin$ is the only nonlinearity that you have to deal with here.

b) Your linearized system should have at least one eigenvalue that corresponds to a growing exponential. If we just do the formal test for controllability by checking the $(A, \vec{b})$ pair for the linearized system, **does it indicate that we could place the closed-loop eigenvalues wherever we want for the linearized system?**

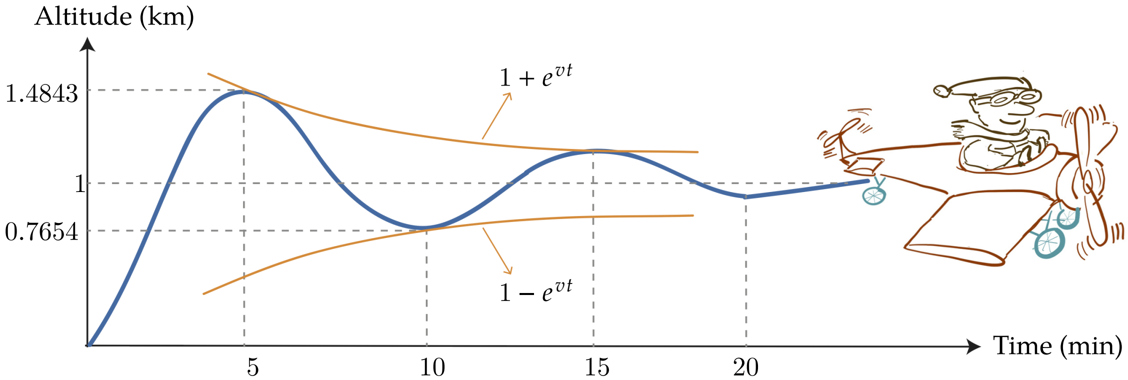
c) Using state feedback, Justin has selected the control gains $K = \begin{bmatrix} 20 & 5 & 0.01 \end{bmatrix}$ for his input $u = K\vec{x}$. **What are the eigenvalues of the closed loop dynamics for the given K?** Feel free to use numpy.

4 Otto the Pilot

Otto has devised a control algorithm, so that his plane climbs to the desired altitude by itself. However, he is having oscillatory transients as shown in the figure. Prof. Arcak told him that if his system has complex eigenvalues

$$\lambda_{1,2} = v \pm j\omega,$$

then his altitude would indeed oscillate with frequency ω about the steady state value, $1\,\mathrm{km}$, and that the time trace of his altitude would be tangent to the curves $1 + e^{vt}$ and $1 - e^{vt}$ near its maxima and minima respectively.



a) Find the real part v and the imaginary part ω from the altitude plot.

b) Let the dynamical model for the altitude be

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix},$$

where $y(t)$ is the deviation of the altitude from the steady state value, $\dot{y}(t)$ is the time derivative of $y(t)$, and a_1 and a_2 are constants. Using your answer to part (a), find what a_1 and a_2 are.

c) Otto can change a_2 by turning a knob. Tell him what value he should pick so that he has a "critically damped" ascent with two real negative eigenvalues at the same location.

5 Stability Analysis: Solving least-squares via gradient descent

In this problem, we will derive a dynamical system approach for solving a least-squares problem which finds the $\vec{x}$ that minimizes $||A\vec{x} - \vec{y}||^2$. We consider A to be tall and full rank.

Recall that the Least Squares problem has a closed-form solution:

$$\vec{\hat{x}} = (A^T A)^{-1} A^T \vec{y}.$$

Direct computation requires the "inversion" of $A^T A$, which has a complexity of $O(N^3)$ where $(A^T A) \in \mathbb{R}^{N \times N}$. This may be okay for small problems with a few parameters, but can easily become unfeasible for large datasets.

Therefore, we will solve this problem iteratively by the gradient descent method, which we can write as a discrete-time state-space system. The Least-Sqaures problem can be expressed as the following optimization problem

$$\min_{\vec{x} \in \mathbb{R}^n} \left\|\vec{e}\right\|^2 \text{ subject to } \vec{e} = A\vec{x} - \vec{y} \tag{1}$$

a) Given an estimate of $\vec{x}(t)$, we can define the least-squares error $\vec{e}(t)$ to be:

$$\vec{e}(t) = \vec{y} - A\vec{x}(t)$$

Show that if $\vec{x}(t) = \vec{\hat{x}}$, then $\vec{e}(t)$ is orthogonal to the columns of A.

b) We would like to develop a "fictionary" state space equation for which the state $\vec{x}(t)$ will converge to $\vec{x}(t) \to \vec{\hat{x}}$, the least squares solution. If the argument $A^T(A\vec{\hat{x}} - \vec{y}) = 0$ then we define the following update:

$$\vec{x}(t+1) = \vec{x}(t) - \alpha A^T(A\vec{x}(t) - \vec{y})$$

You can see when $\vec{x}(t) = \vec{\hat{x}}$, the system reaches equilibrium. By the way, it is no coincidence that the gradient of $||A\vec{x} - \vec{y}||^2$ is

$$\nabla ||A\vec{x} - \vec{y}||^2 = 2A^T(A\vec{x} - \vec{y})$$

This can be derived by vector derivatives (outside of class scope) or by partial derivatives as we did in the linearization case.

To show that $\vec{x} \to \vec{\hat{x}}$, we define a new state variable $\vec{\epsilon}(t) = \vec{x}(t) - \vec{\hat{x}}$

Derive the discrete-time state evolution equation for $\vec{\epsilon}(t)$, and show that it takes the form:

$$\vec{\epsilon}(t+1) = (I - \alpha G)\vec{\epsilon}(t). \tag{2}$$

c) We would like to make the system such that $\vec{\epsilon}(t)$ converges to 0. **Show that the eigenvalues of matrix $I - \alpha G$ are $1 - \alpha\lambda_{\{G\}}$, where $\lambda_{\{G\}}$ are the eigenvalues of G. Also explain why all of the eigenvalues $\lambda_{\{G\}}$ are greater than zero.**

d) **For what α would the eigenvalue $1 - \alpha\lambda_{max\{G\}} = 0$ where $\lambda_{max\{G\}}$ is the largest eigenvalue of G. At this α, what would be the largest magnitude eigenvalue of $I - \alpha G$? Is the system stable?**

e) We can increase the learning rate α to speed up our convergence. However, we must be careful and not set the learning rate to be too high. **Above what value of α would the system** (2) **become unstable?** You may assume that α is real and positive.

6 Controllable Canonical Form and Eigenvalue Placement

Consider a discrete-time linear system below ($\vec{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$, and $B \in \mathbb{R}^n$).

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t)$$

If the system is *controllable*, then there exists a transformation $\vec{z} = T\vec{x}$ (where T is the invertible matrix whose columns are the basis vectors for the new representation of the space) such that in the transformed coordinates, the system is in *controllable canonical form*, which is given by

$$\vec{z}(t+1) = \tilde{A}\vec{z}(t) + \tilde{B}u(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

Here, $\tilde{A} = TAT^{-1}$ and $\tilde{B} = TB$.
The characteristic polynomials of the matrices A and $\tilde{A}$ are the same and given by

$$\det(\lambda I - A) = \det(\lambda I - \tilde{A}) = \lambda^n - a_{n-1}\lambda^{n-1} - a_{n-2}\lambda^{n-2} - \ldots - a_0. \tag{3}$$

a) **Show that A and $\tilde{A}$ have the same eigenvalues.**

b) Let the controllability matrices C and $\tilde{C}$ be $C = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ and $\tilde{C} = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix}$, respectively. **Show that the desired transformation is given by $T = \tilde{C}C^{-1}$.**

Now, consider the specific **controllable** system

$$\vec{x}(t+1) = A\vec{x}(t) + Bu(t) = \begin{bmatrix} -2 & 0 \\ -3 & -1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} u(t) \tag{4}$$

Since the system is controllable, there exists a transformation $\vec{z} = T\vec{x}$ such that

$$\vec{z}(t+1) = \tilde{A}\vec{z}(t) + \tilde{B}u(t) = \begin{bmatrix} 0 & 1 \\ a_0 & a_1 \end{bmatrix} \vec{z}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{5}$$

and the characteristic polynomials of the matrices A and $\tilde{A}$ are the same.

c) **Compute the transformation matrix $T = \tilde{C}C^{-1}$.**

This systems turns out to be unstable. In the controller basis, we would use a feedback control law $u(t) = -\tilde{K}\vec{z}(t) = \begin{bmatrix} -\tilde{k}_0 & -\tilde{k}_1 \end{bmatrix} \vec{z}(t)$ to place eigenvalues and stabilize the system.

d) If we want to apply the same feedback control law directly using the original $\vec{x}(t)$ state, call the resulting law $u(t) = -K\vec{x}(t) = \begin{bmatrix} -k_0 & -k_1 \end{bmatrix} \vec{x}(t)$. **Give an expression for K in terms of $\tilde{K}$ and T.**

e) **Compute $\tilde{K}$ so that $\tilde{A}_{cl}$ has eigenvalues $\lambda = \pm\frac{1}{2}$ Convert $\tilde{K}$ back to the original basis so that we may obtain our feedback control law for the original $\vec{x}(t)$ state.**

7 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **What sources (if any) did you use as you worked through the homework?**

b) **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

c) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

d) **Do you have any feedback on this homework assignment?**

e) **Roughly how many total hours did you work on this homework?**