This homework is due on Tuesday, July 7, 2020, at 11:59PM.
Self-grades are due on Tuesday, July 14, 2020, at 11:59PM.

1 Complex Numbers

A common way to visualize complex numbers is to use the complex plane. Recall that a complex number z is often represented in Cartesian form.

$$z = x + jy \text{ with } \mathrm{Re}\{z\} = x \text{ and } \mathrm{Im}\{z\} = y$$

See Figure 1 for a visualization of z in the complex plane.

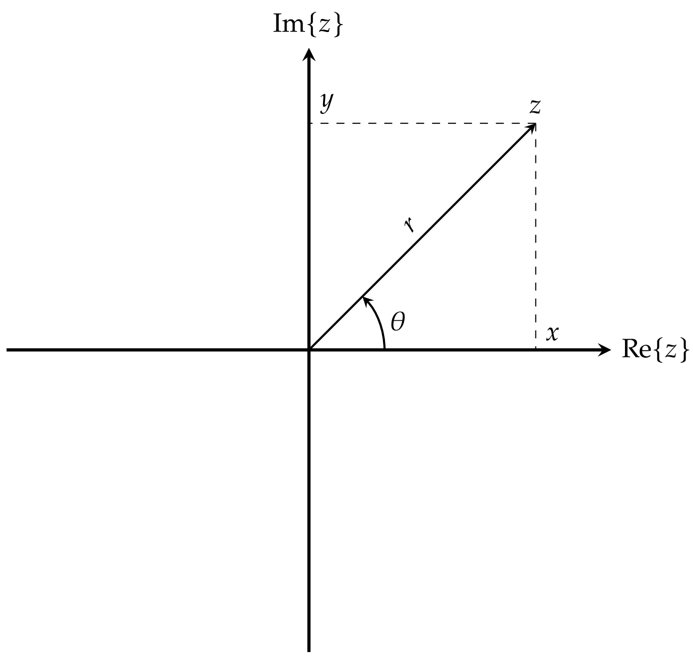


Figure 1: Complex Plane

In this question, we will derive the polar form of a complex number and use this form to make some interesting conclusions.

a) **Calculate the length of z in terms of x and y as shown in Figure 1.** This is the magnitude of a complex number and is denoted by $|z|$ or r. *Hint.* Use the Pythagorean theorem.

b) **Represent x, the real part of z, and y, the imaginary part of z, in terms of r and θ.**

c) **Substitute for x and y in z.** Use Euler's identity[1] $e^{j\theta} = \cos\theta + j\sin\theta$ to **conclude that,**

$$z = re^{j\theta}$$

d) In the complex plane, **draw out all the complex numbers such that $|z| = 1$. What are the z values where the figure intersects the real axis and the imaginary axis?**

[1]also known as de Moivre's Theorem.

e) If $z = re^{j\theta}$, **prove that** $\bar{z} = re^{-j\theta}$. Recall that the complex conjugate of a complex number $z = x + jy$ is $\bar{z} = x - jy$.

f) **Show that,**

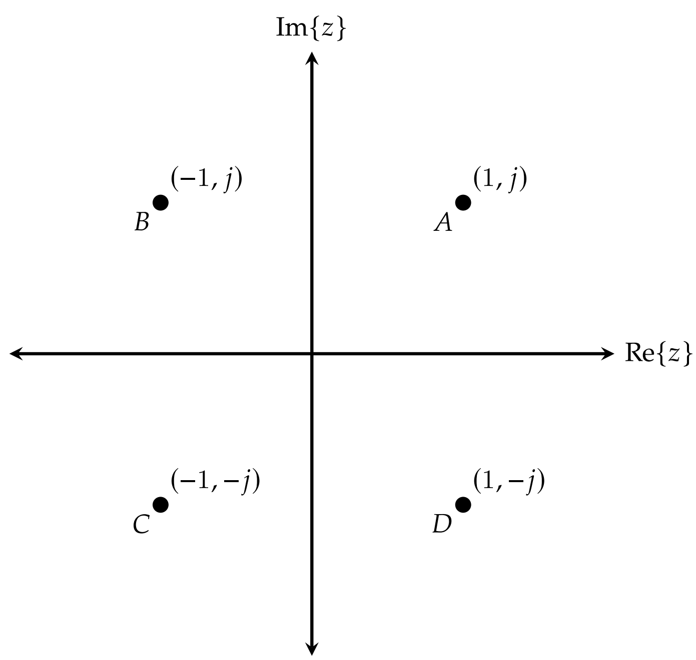
$$r^2 = z\bar{z}$$

g) Intuitively **argue that**

$$\sum_{k=0}^{2} e^{j\frac{2\pi}{3}k} = 0$$

Do so by drawing out the different values of the sum in the complex plane and making an argument based on the vector sum.

h) In modern wireless communication, signals are sent as complex exponentials $e^{j\omega t}$, with receivers detecting both the cosine and sine components of the signal. One common scheme for encoding digital data, such as what is used in your phone and in WiFi, is known as quadrature phase-shift keying (QPSK). In this technique, there are four points of interest on the complex plane (see figure below):

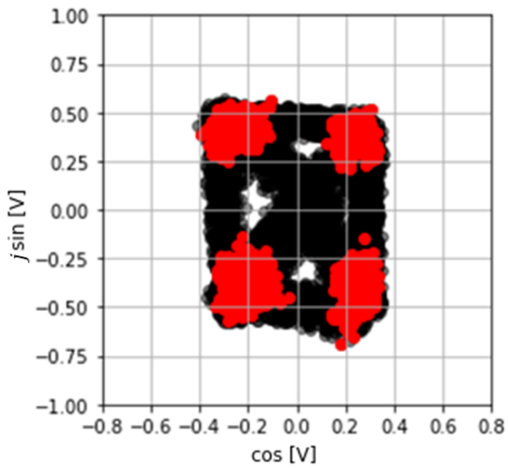


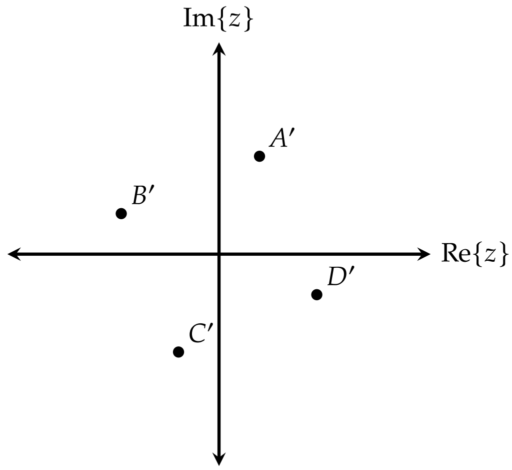
Write each of the points A, B, C, and D in polar form.

i) Due to amplitude error and phase noise, in practice these points often arrive at the receiver scaled and rotated (see the figure on the left). Suppose the received points are as follows:

$$A' = 1e^{j\frac{3\pi}{8}} \qquad C' = 1e^{j\frac{-5\pi}{8}}$$

$$B' = 1e^{j\frac{7\pi}{8}} \qquad D' = 1e^{j\frac{-\pi}{8}}$$



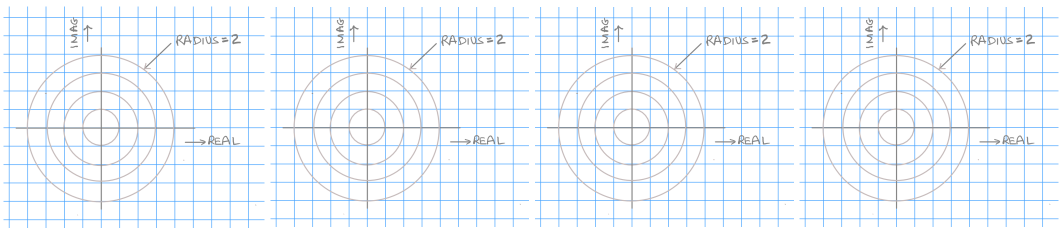


Find a corrective value $r_x e^{j\theta_x}$ that when multiplied with A', B', C', D' recovers the original points A, B, C, D. What is the original noise $r_n e^{j\theta_n}$ that created the shift?

2 Phasors

a) Consider a resistor ($R = 1.5\Omega$), a capacitor ($C = 1\text{F}$), and an inductor ($L = 1\text{H}$) connected in series. **Give expressions for the impedances of Z_R, Z_C, Z_L for each of these elements as a function of the angular frequency ω.**

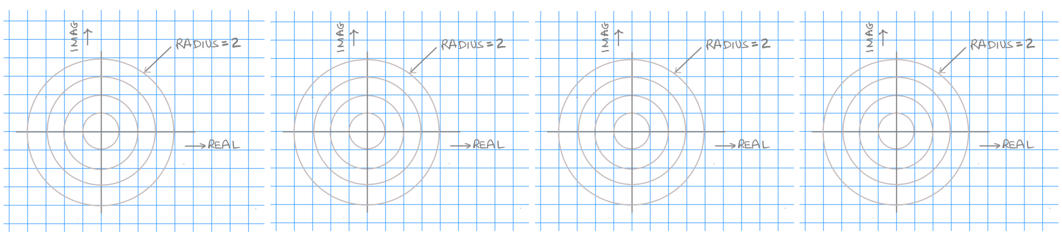
b) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = \frac{1}{2}$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Then give the magnitude and phase of Z_{total}.**



(a) $Z_R(@\omega = 0.5)$ (b) $Z_C(@\omega = 0.5)$ (c) $Z_L(@\omega = 0.5)$ (d) $Z_{total}(@\omega = 0.5)$

Figure 2: Impedances at $\omega = 0.5$.

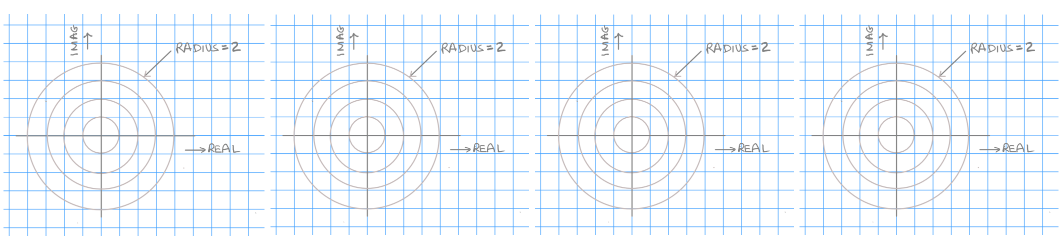
c) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = 1$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Then give the magnitude and phase of Z_{total}.**



(a) $Z_R(@\omega = 1)$ (b) $Z_C(@\omega = 1)$ (c) $Z_L(@\omega = 1)$ (d) $Z_{total}(@\omega = 1)$

Figure 3: Impedances at $\omega = 1$.

d) **Draw the individual impedances as "vectors" on the same complex plane for the case $\omega = 2$ rad/sec. Also draw the combined impedance Z_{total} of their series combination. Then give the magnitude and phase of Z_{total}.**



(a) $Z_R(@\omega = 2)$ (b) $Z_C(@\omega = 2)$ (c) $Z_L(@\omega = 2)$ (d) $Z_{total}(@\omega = 2)$

Figure 4: Impedances at $\omega = 2$.

e) **For the previous series combination of RLC elements, what is the "natural frequency" ω_n where the series impedance is purely real?**

3 Color Organ Filter Design

In the third lab, we will design low-pass and high-pass filters for a color organ. There are red, and green LEDs. Each color will correspond to a specified frequency range of the input audio signal. The intensity of the light emitted will correspond to the amplitude of the audio signal for that frequency range.

a) First, you realize that you can build simple filters using a resistor and a capacitor. Design the first-order **passive** low and high pass filters with following frequency ranges for each filter using $1\,\mu F$ capacitors. ("Passive" means that the circuit components making up the filter do not require any power supply.)

- Low pass filter – 3-dB frequency at $2400\,\mathrm{Hz} = 2\pi \cdot 2400 \frac{rad}{sec}$
- High pass filter – 3-dB frequency at $100\,\mathrm{Hz} = 2\pi \cdot 100 \frac{rad}{sec}$

Draw the schematic-level representation of your designs and show your work finding the resistor values. Also, please mark V_{in}, V_{out}, and ground nodes in your schematic. Round your results to two significant figures. Remember that $\omega = 2\pi f$!

b) Bode plots are a useful tool to understand the frequency response of any filter you might design. First, write out the transfer functions $H_{\text{LPF}} = \frac{V_{\text{out, LPF}}}{V_{\text{in, LPF}}}$ and $H_{\text{HPF}} = \frac{V_{\text{out, HPF}}}{V_{\text{in, HPF}}}$ for both filters designed in part (a). Then, use a computer to draw the Bode **magnitude** plot from $0.1\,\mathrm{Hz}$ to $1\,\mathrm{GHz}$.

If you use Python, plot 10,000 samples of the magnitude of the bode plot. Here are a couple useful functions (*make sure to lookup documentation for each of these!*):

- `scipy.signal.TransferFunction`
- `scipy.signal.bode`
- `np.linspace`
- `matplotlib.pyplot`

c) You decide to build a bandpass filter by simply cascading the first-order low-pass and high-pass filters you designed in part (a). Connect the V_{out} node of your low-pass filter directly to the V_{in} node of your high pass filter. The V_{in} of your new band-pass filter is the V_{in} of your old low-pass filter, and the V_{out} of the new filter is the V_{out} of your old high-pass filter.

What is H_{BPF}, the transfer function of your new band-pass filter? Use R_L, C_L, R_H, and C_H for low-pass filter and high-pass filter components, respectively. Show your work.

d) Plug the component values you found in (a) into the transfer function H_{BPF}. **Then, using a computer via Python or your favorite graphing tool, draw the Bode magnitude plot from $0.1\,\mathrm{Hz}$ to $1\,\mathrm{GHz}$.**

What is the maximum magnitude of H_{BPF}? Is that something that you want? If not, explain why not and suggest a simple way (either adding passive or active components) to fix it.

4 Transfer Functions and Filters

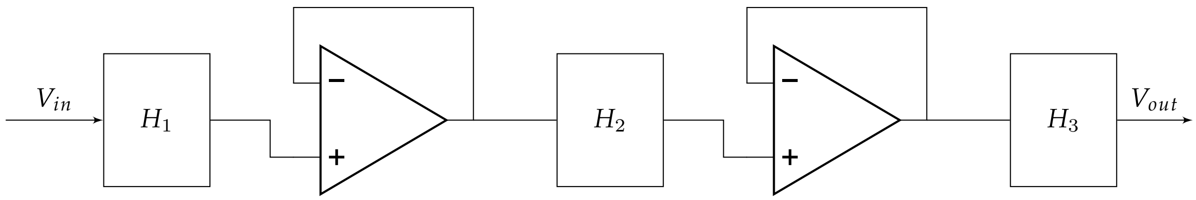


Figure 5: Three filters cascaded via unity-gain op-amp buffers

Suppose that at some frequency ω_0 radians/sec we know that:

$$H_1(j\omega_0) = 3e^{j\frac{\pi}{4}} \quad H_2(j\omega_0) = \frac{1}{2}e^{-j\frac{\pi}{3}} \quad H_3(j\omega_0) = 4e^{j\frac{5\pi}{6}}$$

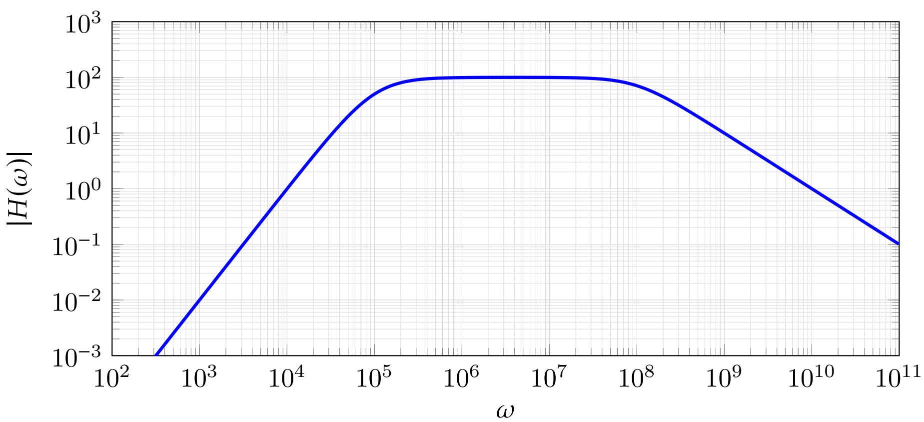
If $V_{in}(t) = 2\sin\left(\omega_0 t + \frac{\pi}{2}\right)$:

a) What is the phasor $\widetilde{V}_{in}$ of the input voltage?

b) What is the phasor $\widetilde{V}_{out}$ of the output voltage?

c) What is $V_{out}(t)$?

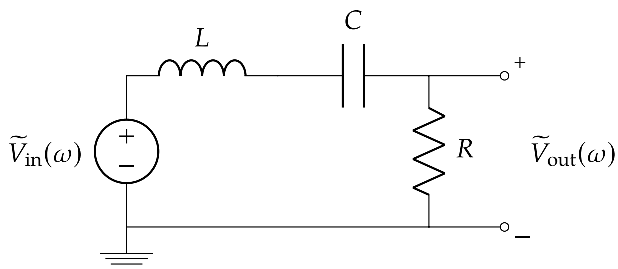
d) The Bode magnitude plot of the $H(\omega)$ is the following.



What is $H(\omega)$?

5 Band-pass filter

It is quite common to need to design a filter which selects only a narrow range of frequencies. One example is in WiFi radios, it is desirable to select only the 2.4GHz frequency containing your data, and reject information from other nearby cellular or bluetooth frequencies. This type of filter is called a band-pass filter; we will explore the design of this type of filter in this problem.



a) **Write down the impedance of the series RLC combination in the form $Z_{RLC}(\omega) = A(\omega) + jX(\omega)$, where $X(\omega)$ is a real valued function of ω.**

b) **Write down the transfer function $H(\omega) = \frac{\widetilde{V}_{\text{out}}(\omega)}{\widetilde{V}_{\text{in}}(\omega)}$ for this circuit.**

c) **At what frequency ω_n does $X(\omega_n) = 0$?** (i.e. at what frequency is the impedance of the series combination of RLC purely real — meaning that the imaginary terms coming from the capacitor and inductor completely cancel each other.)

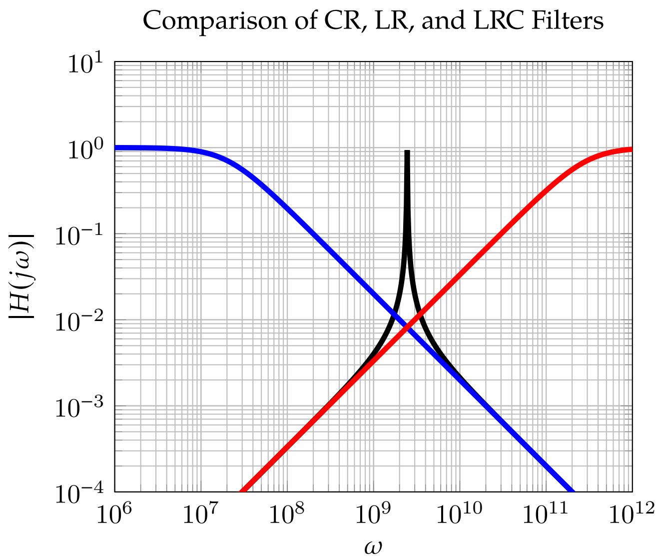
   What happens to the relative magnitude of the impedances of the capacitor and inductor as ω moves above and below ω_n? What is the value of the transfer function at this frequency?

d) In most filters, we are interested in the cutoff frequency, since that helps define the frequency range over which the filter operates. Remember that this is the frequency at which the magnitude of the transfer function drops by a factor of $\sqrt{2}$ from its maximum value. Notice that the real part of the impedance Z_{RLC} is not changing with frequency and stays at R. What we care about is the frequencies where the imaginary part of the impedance equals $-jR$ to $+jR$.

   Solve for the cutoff frequencies by finding ω_{c1} and ω_{c2} where the imaginary part of the impedance $Z_{RLC} = \pm R$. Then use these cutoff frequencies to calculate the bandwidth $\Delta\omega = |\omega_{c1} - \omega_{c2}|$.

e) **Simplify $X(\omega)$ in two cases, when $\omega \to \infty$ and when $\omega \to 0$. Plug this simplified $X(\omega)$ into your previously solved expressions to find the transfer function at high and low frequencies.**

f) **A bode plot for a CR filter, a LR filter, and a LCR filter is shown for $L = 9nH$, $R = 0.18\Omega$, and $C = 18.7pF$. Assign each filter to its corresponding line on the plot. Label the locations of the corner frequencies, and describe the behavior of the LCR filter at very high and low frequencies.**


Comparison of CR, LR, and LRC Filters
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$|H(j\omega)|$
10^6
10^7
10^8
10^9
10^{10}
10^{11}
10^{12}
ω

6 Matrix Differential Equations

In this problem, we consider ordinary differential equations which can be written in the following form

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \tag{1}$$

where x, y are variables depending on t, $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$, and A is a 2×2 matrix with constant coefficients. We call (1) a matrix differential equation.

a) Suppose we have a system of ordinary differential equations

$$x' = 7x - 8y \tag{2}$$
$$y' = 4x - 5y \tag{3}$$

Write this in the form of (1).

b) Compute the eigenvalues of the matrix A from the previous part.

c) We claim that the solution for $x(t), y(t)$ is of the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_0 t} + c_3 e^{\lambda_1 t} \end{bmatrix},$$

where c_0, c_1, c_2, c_3 are constants, and λ_0, λ_1 are the eigenvalues of A. Suppose that the initial conditions are $x(0) = 1$, $y(0) = -1$. Solve for the constants c_0, c_1, c_2, c_3.

Hint: What are $\frac{dx}{dt}(0)$ *and* $\frac{dy}{dt}(0)$?

d) Verify that the solution for $x(t), y(t)$ found in the previous part satisfies the original system of differential equations (2), (3).

e) We now apply the method above to solve another second-order ordinary differential equation. Suppose we have the system

$$z''(t) - 5z'(t) + 6z(t) = 0, \tag{4}$$

where $z' = \frac{dz}{dt}$ and $z'' = \frac{d^2z}{dt^2}$. Write this in the form of (1), by choosing your state variables to be $x(t) = z(t)$, $y(t) = z'(t)$.

f) Solve the system in (4) with the initial conditions $z(0) = 1$, $z'(0) = 1$, using the method developed in parts (b) and (c).

7 Homework Process and Study Group

Citing sources and collaborators are an important part of life, including being a student! We also want to understand what resources you find helpful and how much time homework is taking, so we can change things in the future if possible.

a) **Watch any of the 16A review discussions (1A or 1B, found here), and explain one of the following topics in your own words: passive sign convention, nodal analysis, superposition, Thevenin and Norton equivalent circuits, or op-amps in negative feedback. More 16A circuits resources can be found at `https://www.eecs16a.org`, notes 11-20.**

b) **What sources (if any) did you use as you worked through the homework?**

c) **If you worked with someone on this homework, who did you work with?** List names and student ID's. (In case of homework party, you can also just describe the group.)

d) **How did you work on this homework?** (For example, *I first worked by myself for 2 hours, but got stuck on problem 3, so I went to office hours. Then I went to homework party for a few hours, where I finished the homework.*)

e) **Do you have any feedback on this homework assignment?**

f) **Roughly how many total hours did you work on this homework?**