EECS 16B Summer 2020 Midterm 2 (Form 1)

Instructions

You have 90 minutes. You may access printed, handwritten, and pre-downloaded materials during this exam. You may not consult the Internet or other human beings during this exam. You may not share the contents of this exam by any medium, verbal or digital, at any time before it is posted on the course website.

Justify your answers unless otherwise stated. A correct result without justification will not receive full credit.

Honor Code

COPY, SIGN, and DATE the UC Berkeley Honor Code by hand. (*If you fail to do so, your exam may not be accepted.* Email the instructors if you are unable.)

> *As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.*

. . . .

COPY, SIGN, and DATE the following addendum by hand. (*If you fail to do so, your exam may not be accepted.* Email the instructors if you are unable.)

> *Should I commit academic misconduct during this exam, let me receive a failing grade in EECS 16B or dismissal from the University.*

. . . .

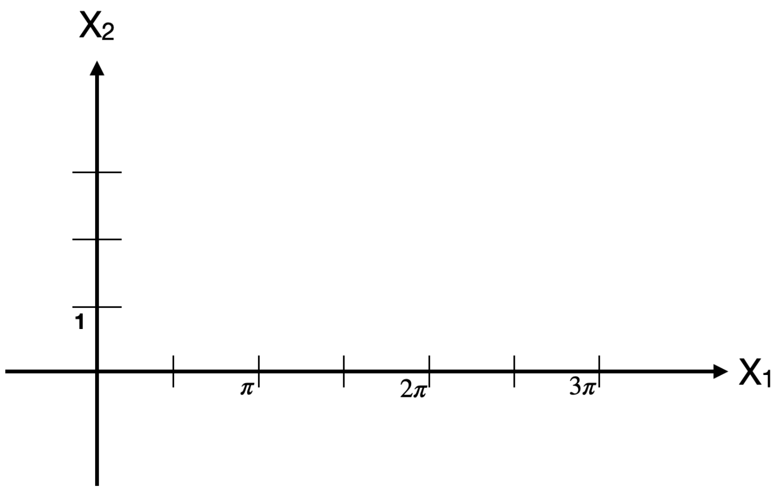
1 Equilibria

a) Consider the following non-linear, continuous-time system:

$$\dot{x}(t) = \begin{bmatrix} f_1(x_1(t), x_2(t)) \\ f_2(x_1(t), x_2(t)) \end{bmatrix} = \begin{bmatrix} x_2(t) - sin(x_1(t)) - 1 \\ x_2(t) - 2e^{-x_1(t)} \end{bmatrix}$$

where $x(t) \in \mathbb{R}^2$.

i) Plot the conditions satisfying $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$ on the same plot. Don't worry about finding the exact values for which the two curves intersect, so long as your plot is qualitatively accurate.



ii) How many equilibrium points does this system have? Explain.

b) Consider the following non-linear, discrete-time system:

$$x[k+1] = \begin{bmatrix} f_1(x_1[k], x_2[k]) \\ f_2(x_1[k], x_2[k]) \end{bmatrix} = \begin{bmatrix} sin(\frac{\pi}{2} x_1[k]) \\ 1 - x_1[k] + x_2[k]^2 \end{bmatrix}$$

where $x[k] \in \mathbb{R}^2$. What are the equilibria of this system?

2 Stability

a) Consider the system

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

with $x(t) \in \mathbb{R}^2$. Is this system stable? Explain.

b) Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

with $x(t) \in \mathbb{R}^2$. Is this system stable? Explain.

c) Consider the system

$$x_{k+1} = \begin{bmatrix} -0.25 + j & 0 \\ 0 & -0.25 - j \end{bmatrix} x_k + \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

with $x_k \in \mathbb{C}^2$. Is this system stable? Explain.

3 Controllability

a) Consider the system

$$x_{k+1} = \begin{bmatrix} 1 & 0 & 0.1 \\ 0 & 1 & 0.1 \\ 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} u_k$$

with $x_k \in \mathbb{R}^3$, and $u_k \in \mathbb{R}$. Is this system controllable? Explain.

b) Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

with $x(t) \in \mathbb{R}^4$, and $u(t) \in \mathbb{R}$. Is this system controllable? Explain.

Hint: Can you determine this without constructing the controllability matrix?

c) Consider the system

$$\dot{x}(t) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} u(t)$$

with $x(t) \in \mathbb{R}^{10}$, and $u(t) \in \mathbb{R}$. Is this system controllable? Explain.

Hint: Can you determine this without constructing the controllability matrix?

4 SVD

a) How can the SVD of a matrix A be used to find a basis for the null-space of A? Justify your answer (1-2 sentences).

b) How can the SVD of a matrix A be used to find a basis for the column-space of A? Justify your answer (1-2 sentences).

5 Optimal Control

Consider a model of a system given by the following:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

Here, $x(t) \in \mathbb{R}^4$ and $u(t) \in \mathbb{R}^2$.

a) Consider a discretized model of this system of the form

$$x[k+1] = A_d x[k] + B_d u[k],$$

where $x[k] := x(Tk)$, T is the discretization sampling period,

$$u(t) = u[k] : t \in [Tk, T(k+1)),$$

and A_d and B_d are given by

$$A_d := \begin{bmatrix} 1 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_d := \begin{bmatrix} 0.125 & 0 \\ 0.5 & 0 \\ 0 & 0.125 \\ 0 & 0.5 \end{bmatrix}.$$

Find the value of T used in this diescretization. Show your work.

b) Is the **discrete-time** system from part (a) controllable? Explain.

c) Let $x[0] := \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top$. Find the minimum value k such that the discrete-time system in part (a) can reach the *particular* goal

$$x[k] = x_{goal} := \begin{bmatrix} 0.5 \\ 2 \\ -2 \\ -8 \end{bmatrix}$$

What are the corresponding controls which achieve this goal?

d) Let $x[0] := \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top$. We wish to control the system to the configuration $x[3] = \begin{bmatrix} 4 & 0 & 2 & 0 \end{bmatrix}^\top$, such that the quantity

$$\|u[0]\|_2^2 + \|u[1]\|_2^2 + \|u[2]\|_2^2$$

is minimized. Express the solution to this problem in the form

$$\begin{bmatrix} u^*[2] \\ u^*[1] \\ u^*[0] \end{bmatrix} = G \begin{bmatrix} 4 \\ 0 \\ 2 \\ 0 \end{bmatrix},$$

for some matrix G. When forming G, you may express it symbolically in terms of the matrices A_d and B_d.

6 System Identification

Consider a discrete-time system with unknown dynamics. Assume that starting from $x_0 = \begin{bmatrix} 1 & 2 \end{bmatrix}^\top$ we applied the following controls to the system, and observed the resulting states:

$$u_0 = 1, \quad u_1 = 2, \quad u_2 = 0, \quad u_3 = 1,$$
$$x_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, x_2 = \begin{bmatrix} 4 \\ 8 \end{bmatrix}, x_3 = \begin{bmatrix} 8 \\ 1 \end{bmatrix}, x_4 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

a) Set up a least-squares problem to recover $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^{2\times 1}$ of a discrete-time model of this system

$$x_{k+1} = Ax_k + Bu_k$$

There is no need to solve for the A and B matrices.

b) Could the estimates of A and B be uniqueley determined from less observations than those given? Explain. .

7 Feedback Control

Consider the system

$$x[k+1] = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[k]$$

where $x[k] \in \mathbb{R}^2$ and $u[k] \in \mathbb{R}$.

a) Assume $u[k]$ is chosen at each time k by a feedback controller given by the matrix K, i.e.

$$u[k] := -Kx[k]$$

where $K \in \mathbb{R}^{1\times 2}$. Express the closed-loop model of this system in the form

$$x[k+1] = \hat{A}x[k]$$

for some matrix $\hat{A}$ to find. .

b) Find the matrix K such that the closed-loop system from part (a) has eigenvalues $\{0.5, -0.5\}$.