Circulant Matrices

A square matrix C_h is circulant if each row vector is rotated one element to the right relative to the preceding row vector.

$$C_h = \begin{bmatrix} h_0 & h_{N-1} & \cdots & h_2 & h_1 \\ h_1 & h_0 & h_{N-1} & & h_2 \\ \vdots & h_1 & h_0 & \ddots & \vdots \\ h_{N-2} & \vdots & \ddots & \ddots & h_{N-1} \\ h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{bmatrix} \tag{1}$$

Recall from lecture that we can describe the input-output relationship of a periodic discrete-time LTI system via a circulant matrix.

$$\vec{y} = C_h \vec{x} \tag{2}$$

In this case, the first column of C_h is the impulse response $h[n]$ of the system.

$$\vec{h} = \begin{bmatrix} h_0 & h_1 & \cdots & h_{N-2} & h_{N-1} \end{bmatrix} \tag{3}$$

Rather beautifully, the DFT basis vectors are eigenvectors of C_h. We will have N DFT vectors, since that is the dimensionality of our model.

$$\vec{u_k} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & e^{j\frac{2\pi}{N}k\cdot 1} & \cdots & e^{j\frac{2\pi}{N}k\cdot(N-1)} \end{bmatrix} \tag{4}$$

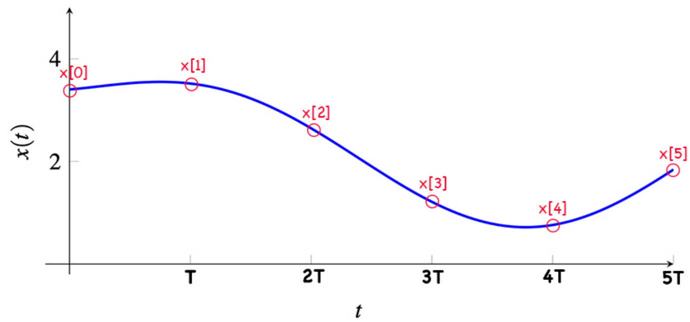
Letting $H[k]$ be the k^{th} DFT coefficient of $h[n]$, we can write the following eigenvalue equation for $k = 0, 1, \cdots, N-1$.

$$C_h \vec{u}_k = \underbrace{\left(\sqrt{N} \times H[k]\right)}_{\text{eigenvalue}} \vec{u}_k \tag{5}$$

In this discussion you'll see why this is useful by representing convolution as a circulant matrix C_h, and then diagonalizing it. This will draw the connection between the DFT and LTI systems.

Sampling theorem

Let x be continuous signal bandlimited by frequency ω_{max}. We sample x with a period of T_s.



Given the discrete samples, we can try reconstructing the original signal f through sinc-interpolation where $\Phi(t) = \text{sinc}\left(\frac{t}{T_s}\right)$

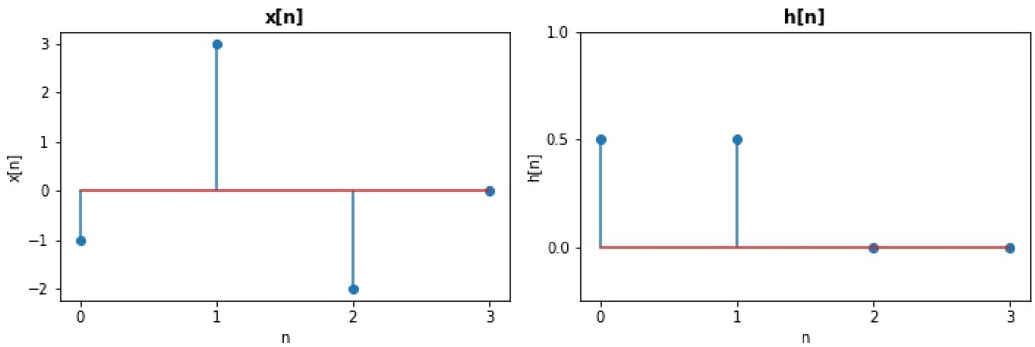
$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x[n]\Phi(t - nT_s)$$

We define the **sampling frequency** as $\omega_s = \frac{2\pi}{T_s}$. The Sampling Theorem says if $\omega_{max} < \frac{\pi}{T_s}$, or $\omega_s > 2\omega_{max}$, then we are able to recover the original signal, i.e. $x = \hat{x}$.

1 Circulant Matrices & Convolution

Consider the signal $x[n]$ of length 3 and an impulse response $h[n]$ of length 2. You may assume that they are zero everywhere else.

$$\vec{x} = \begin{bmatrix} -1 & 3 & -2 \end{bmatrix}^T \qquad \vec{h} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T \tag{6}$$



a) What is the convolution $y[n] = x[n] * h[n]$? Also what is the length of this output signal?

b) Now write each term of the output signal $y[n]$ as a sum using the convolution formula and set up a matrix equation $\vec{y} = A\vec{x}$. What is the size of this matrix?

c) Add elements to the matrix A and zeros to the vector $\vec{x}$ to create a square matrix C_h that is circulant.

d) Since the DFT diagonalizes circulant matrices, lets try to solve for the output signal $y[n]$ using the DFT instead of convolution.

- Step 1: Compute the DFT of $x[n]$ and $h[n]$: $\vec{X} = F\vec{x}$, $\vec{H} = F\vec{h}$.
- Step 2: Take the elementwise product of the DFTs and scale: $\vec{Y} = \sqrt{N}\vec{X} \odot \vec{H}$.
- Step 3: Perform the inverse DFT to get the result $\vec{y} = F^*\vec{Y}$.

e) What is the importance behind this result? Compare the runtimes between convolution and the Fast Fourier Transform (FFT) which takes $O(N \log N)$ operations.

2 Sampling Theorem basics

Consider the following signal, $x(t)$ defined as,

$$x(t) = \cos(2\pi t)$$

a) Find the maximum frequency, ω_{max}, in radians per second? In Hertz? (From now on, frequencies will refer to radians per second.)

b) If I sample every T seconds, what is the sampling frequency?

c) What is the smallest sampling period T that would result in an imperfect reconstruction?

3 More Sampling

Let's sample the signal from the previous question x with sampling period $T_m = \frac{1}{4}$s and $T_n = 1$s and perform sinc interpolation on the resulting samples. Let the reconstructed functions be f_m and f_n.

a) Have we satisfied the Nyquist limit (i.e. the sampling theorem) in any case?

b) What is the highest frequency we can reconstruct with the sampling rate T_n?

c) Based on this answer, can you think of any periodic function that has a frequencies less than or equal to π that samples the same as f_n?

4 Aliasing

Consider the signal $x(t) = \sin(0.2\pi t)$.

a) At what period T should we sample so that sinc interpolation recovers a function that is identically zero?

b) At what period T can we sample at so that sinc interpolation recovers the function $f(t) = -\sin\left(\frac{\pi}{15}t\right)$?