1 Discrete Time Systems

Consider a discrete-time system with $x[n]$ as input and $y[n]$ as output.

$$x[n] \longrightarrow \boxed{} \longrightarrow y[n]$$

The following are some of the possible properties that a system can have:

Linearity

A **linear system** has the properties below:

1. **additivity**

$$x_1[n] + x_2[n] \longrightarrow \boxed{} \longrightarrow y_1[n] + y_2[n] \tag{1}$$

2. **scaling**

$$\alpha x[n] \longrightarrow \boxed{} \longrightarrow \alpha y[n] \tag{2}$$

Here, α is some constant.
Together, these two properties are known as **superposition**:

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \longrightarrow \boxed{} \longrightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

Time Invariance

A system is **time-invariant** if its behavior is fixed over time:

$$x[n - n_0] \longrightarrow \boxed{} \longrightarrow y[n - n_0] \tag{3}$$

Causality

A **causal** system has the property that $y[n_0]$ only depends on $x[n]$ for $n \in (-\infty, n_0]$. An intuitive way of interpreting this condition is that the system does not "look ahead."

Bounded-Input, Bounded-Output (BIBO) Stability

In a BIBO stable system, if $x[n]$ is bounded, then $y[n]$ is also bounded. A signal $x[n]$ is bounded if there exists an M such that $|x[n]| \leq M < \infty \; \forall n$.

2 Linear Time-Invariant (LTI) Systems

A system is LTI if it is both linear and time-invariant. We define the **impulse response** of an LTI system as the output $h[n]$ when the input $x[n] = \delta[n]$ where $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$.

An LTI system can be uniquely characterized by its impulse response $h[n]$. In addition, the following properties hold:

- An LTI system is causal iff $h[n] = 0 \ \forall n < 0$.
- An LTI system is BIBO stable iff its impulse reponse is absolutely summable:

$$\sum_{n=-\infty}^{\infty} \left|h[n]\right| < \infty$$

Convolution Sum

Consider the following LTI system with impulse reponse $h[n]$:

$$x[n] \longrightarrow \boxed{} \longrightarrow y[n]$$

Notice that we can write $x[n]$ as a sum of impulses:

$$x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

In addition, we know that:

$$\delta[n] \longrightarrow \boxed{} \longrightarrow h[n]$$

By applying the LTI property of our system, we get that

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The expression $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$ is referred to as the **convolution sum** and can be written as $x[n] * h[n]$ or $(x * h)[n]$.

3 Is it LTI?

Determine if the following systems are LTI:

a) $y[n] = 4x[n]$

b) $y[n] = 2x[n] - 4$

c) $y[n] = 2x[-2 + 3n] + 2x[2 + 3n]$

d) $y[n] = 4^{x[n]}$

e) $y[n] - y[n-1] = x[n]$

f) $y[n] = x[n] + nx[n-1]$

4 Convoluted Convolution

a) Show that convolution is commutative. That is, show that $(x * h)[n] = (h * x)[n]$.

b) Show that $\delta[n]$ is a convolution identity. That is, show that $(x * \delta)[n] = x[n]$.

c) Show that convolution by $\delta[n - n_0]$ shifts $x[n]$ by n_0 steps to the right.

d) Show that convolution is distributive. In other words, show that $(x * (h_1 + h_2))[n] = (x * h_1)[n] + (x * h_2)[n]$.

5 Mystery System

Consider an LTI system with the following impulse response:

$$h[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$$

a) Create a sketch of this impulse reponse.

b) What is the output of our system if the input is the unit step $u[n]$?

c) What is the output of our system if our input is $x[n] = (-1)^n u[n]$?

d) This system is called the two-point simple moving average (SMA) filter. Based on the previous parts, why do you think it bears this name?