Discrete Fourier Transform

Assume we are working with an N length discrete signal and we would like to find its discrete frequencies. This is done through the Discrete Fourier Transform (DFT), which is simply a change of basis to the DFT basis.

First, let us vectorize our signal. If $x[n]$ is our input signal, we model it as a vector by letting the n^{th} coordinate be $x[n]$. In other words,

$$\vec{x} = \left[x[0], x[1], x[2], \dots, x[N-1]\right]^T$$

In order to decompose $\vec{x}$ into its constituent frequencies, we must find the vector representation of these frequencies. A length N signal will have N different discrete frequencies of the following form. The fundamental frequency of the signal is called ω_N and its value is

$$\omega_N = e^{j\frac{2\pi}{N}} \tag{1}$$

The DFT Basis is built by taking powers of this fundamental frequency. We define the k^{th} basis vector $\vec{u}_k[n]$ as

$$\vec{u}_k[n] = \frac{1}{\sqrt{N}}\omega_N^{kn} \text{ for } k = 0, 1, \dots N-1 \tag{2}$$

The matrix U has columns which consist of the N DFT basis vectors

$$U = \begin{bmatrix}\vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1}\end{bmatrix} \tag{3}$$

We choose to normalize all of these vectors by a factor of $\frac{1}{\sqrt{N}}$ so that the DFT basis vectors are orthonormal. Try to verify on your own that

$$\langle \vec{u}_p, \vec{u}_q \rangle = \sum_{n=0}^{N-1} \overline{\vec{u}_q}\vec{u}_p = \begin{cases} 0, & p \neq q \\ 1, & p = q \end{cases}$$

To represent a signal $x[n]$ in the frequency domain, we can change coordinates to the U basis. We define the matrix F as the matrix that takes our time-domain signal and transforms it into the frequency domain.

$$F = U^{-1} = U^* = \frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^{-1\cdot 1} & \omega_N^{-1\cdot 2} & \cdots & \omega_N^{-1\cdot(N-1)} \\ 1 & \omega_N^{-2\cdot 1} & \omega_N^{-2\cdot 2} & \cdots & \omega_N^{-2\cdot(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{-(N-1)\cdot 1} & \omega_N^{-(N-1)\cdot 2} & \cdots & \omega_N^{-(N-1)\cdot(N-1)} \end{bmatrix} \tag{4}$$

Similarly, the matrix U takes a signal $X[k]$ in the frequency domain and converts it back to the time-domain.

$$U = F^{-1} = F^* = \frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^{1\cdot 1} & \omega_N^{1\cdot 2} & \cdots & \omega_N^{1\cdot(N-1)} \\ 1 & \omega_N^{2\cdot 1} & \omega_N^{2\cdot 2} & \cdots & \omega_N^{2\cdot(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{(N-1)\cdot 1} & \omega_N^{(N-1)\cdot 2} & \cdots & \omega_N^{(N-1)\cdot(N-1)} \end{bmatrix} \tag{5}$$

The relationship between a time-domain signal $x[n]$ and its frequency components $X[k]$ can be written as

$$x[n] = X[0]\vec{u}_0 + \dots + X[N-1]\vec{u}_{N-1} = UX[k] \tag{6}$$

1 Roots of Unity

The DFT is a coordinate transformation to a basis made up of roots of unity. In this problem we explore some properties of the roots of unity. An Nth root of unity is a complex number z satisfying the equation $z^N = 1$ (or equivalently $z^N - 1 = 0$).

a) Show that $z^N - 1$ factors as

$$z^N - 1 = (z-1)(\sum_{k=0}^{N-1} z^k).$$

b) Show that any complex number of the form $\omega_k = e^{j\frac{2\pi}{N}k}$ for $k \in \mathbb{Z}$ is an N-th root of unity.

c) Draw the fifth roots of unity in the complex plane. How many of them are there?

d) Let $\omega_1 = e^{j\frac{2\pi}{5}}$. What is ω_1^2? What is ω_1^3? What is ω_1^{42}?

e) What is the complex conjugate of ω_1? What is the complex conjugate of ω_{42}?

f) Compute $\sum_{k=0}^{N-1} \omega^k$ where ω is some root of unity. Does the answer make sense in terms of the plot you drew?

2 DFT of pure sinusoids

a) Consider the continuous-time signal $x(t) = \cos\left(\frac{2\pi}{3}t\right)$. Suppose that we sampled it every 1 second to get (for $n = 3$ time steps):

$$x[n] = \begin{bmatrix} \cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right) \end{bmatrix}^T.$$

Compute $\vec{X}[k]$ and the basis vectors $\vec{u}_k$ for this signal.

b) Now for the same signal as before, suppose that we took $n = 6$ samples. In this case we would have:

$$x[n] = \begin{bmatrix} \cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right) & \cos\left(\frac{2\pi}{3}(3)\right) & \cos\left(\frac{2\pi}{3}(4)\right) & \cos\left(\frac{2\pi}{3}(5)\right) \end{bmatrix}^T.$$

Repeat what you did above. What are $X[k]$ and the basis vectors $\vec{u}_k$ for this signal.

c) Let's do this more generally. For the signal $x(t) = \cos\left(\frac{2\pi m}{N}t\right)$, where m is an integer between 0 and $N-1$, compute the frequency components $X[k]$ where $x[n]$ is a time-domain signal of length N.

$$x[n] = \begin{bmatrix} \cos\left(\frac{2\pi m}{N}(0)\right) & \cos\left(\frac{2\pi m}{N}(1)\right) & \cdots & \cos\left(\frac{2\pi m}{N}(N-1)\right) \end{bmatrix}^T.$$