1 Airplane Discretization

In this question we will explore briefly a simplified linear model of an airplane control model. First let's define the variables we will be working with, in reference with figure 1:

(i) α = Angle of Attack (angle the plane makes with the direction of wind)

(ii) θ = Pitch Angle (angle of the plane with respect to the horizontal)

(iii) u = Elevator angle (used to control the aircraft's pitch)

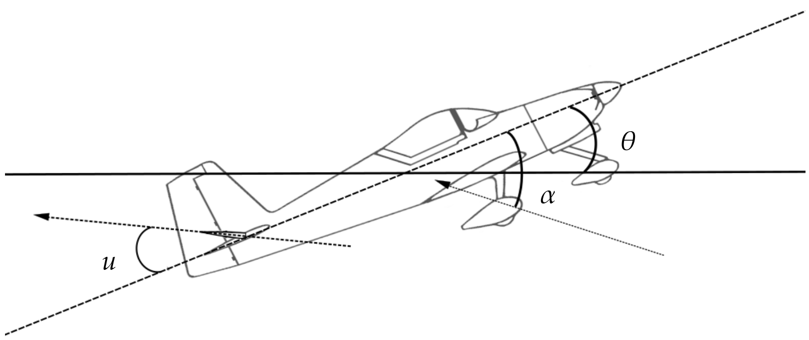


Figure 1: Simplified Airplane model

Now, consider the following (simplified) continuous time system:

$$\frac{d}{dt}\alpha = 5\alpha - \frac{d}{dt}\theta + c_1\delta \tag{1}$$

$$\frac{d^2}{dt^2}\theta = -\alpha + \frac{d}{dt}\theta + c_2\delta \tag{2}$$

a) For the system of differential equations given, **write the matrix differential equation as**

$$\frac{d}{dt}\vec{x} = A\vec{x} + B\delta$$

b) Now, assume for some specific component values we get the following differential equation:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\vec{x}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix}u(t). \tag{3}$$

Unfortunately, we are unable to measure our state vector continuously. Suppose that we sample the system with some sampling interval T. Let us discretize the above system. Assume that we use piecewise constant voltage inputs $u(t) = u[k]$ for $t \in [kT, (k+1)T)$.

Calculate a discrete-time system for Equation (3)'s continuous-time vector system in the form:

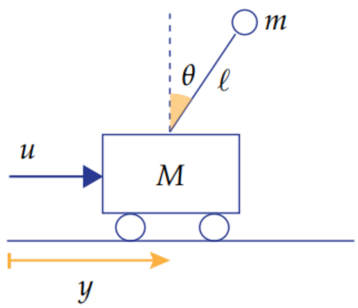
$$\vec{x}[k+1] = A_d\vec{x}[k] + \vec{b}_d[k].$$

2 Inverted Pendulum on a Rolling Cart

Recall the inverted pendulum depicted below from Homework 6 problem 1, which is placed on a rolling cart and whose equations of motion are given by:

$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2\ell\sin\theta - g\sin\theta\cos\theta\right)$$
$$\ddot{\theta} = \frac{1}{\ell(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2\ell\cos\theta\sin\theta + \frac{M+m}{m}g\sin\theta\right).$$

where we use $\dot{x}$ to denote the time derivative of x; that is, $\dot{y} = \frac{dy}{dt}$, $\dot{\theta} = \frac{d\theta}{dt}$, $\ddot{y} = \frac{d^2y}{dt^2}$ and $\ddot{\theta} = \frac{d^2\theta}{dt^2}$.



a) Recall the result of our linearized model from Homework 6, problem 1c:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \approx \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ \frac{M+m}{lM}g & 0 & 0 \\ -\frac{m}{M}g & 0 & 0 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -\frac{1}{lM} \\ \frac{1}{M} \end{bmatrix}}_{B} u.$$

Show that the linearized model is controllable.

b) Suppose $M = 1$, $m = 0.1$, $l = 1$, and $g = 10$, and design a state feedback controller,

$$u(t) = -k_1\theta(t) - k_2\dot{\theta}(t) - k_3\dot{y}(t),$$

such that the eigenvalues of $A - BK$ (the “closed-loop eigenvalues") are $\lambda_1 = \lambda_2 = \lambda_3 = -1$.

c) Suppose we set $k_2 = k_3 = 0$ and vary only k_1; that is, the controller uses only $\theta(t)$ for feedback. Does there exist a k_1 value such that all closed-loop eigenvalues have negative real parts?

3 Minimum Norm Control

Suppose we had a linear discrete-time system with the following dynamics

$$\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{4}$$

Given the initial state $\vec{x}[0] = \vec{0}$, we would like to reach a target state $\vec{x}_t = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

a) Show that we can reach the state $\vec{x}_t$ in a finite number of time-steps

b) What sequence of control inputs can we give to reach the state $\vec{x}_t$ in two time-steps?

c) While we can theoretically reach $\vec{x}_t$ in two time-steps, we notice that it is too expensive to move our system this quickly. Set up an optimization problem of the form below to reach $\vec{x}_t$ in five time-steps with minimum energy.

$$\min_{\vec{w} \in \mathbb{R}^5} \left\|\vec{w}\right\|^2 \text{ subject to } H\vec{w} = \vec{y} \tag{5}$$

d) What is the solution to the optimization problem from the previous part? Give the optimal solution $\vec{w}^*$.

e) Try solving the minimum norm problem for 2 steps all the way up to 10 steps and compare the norms of each solution.