1 System Identification and Linear Control

A scalar discrete-time system has the following dynamics:

$$x(t+1) = \lambda x[t] + g(u[t]),$$

where $g : \mathbb{R} \to \mathbb{R}$ not necessarily linear.

a) If g is approximated to order 2 around the operating point $u^* = 0$, so that

$$x(t+1) \approx \lambda x[t] + \beta_0 + \beta_1 u[t] + \beta_2 u^2[t],$$

what should β_0, β_1, and β_2 be?

b) Suppose that $x[0] = 0$. We apply a sequence of inputs

$$\vec{u} = \big(u[0], u[1], \ldots, u(N-1)\big)$$

and observe states $x[1], x[2], \ldots, x[N]$. Derive the least-squares estimates of λ, β_0, β_1, and β_2.

2 System Identification

Let's now look at how System Identification works in the vector case. Again you are given an unknown discrete-time system. We don't know its specifics but we know that it takes one scalar input and as two observable states.
We would like to find a linear model of the form

$$\vec{x}[t+1] = A\vec{x}[t] + Bu[t] + \vec{w}[t],$$

where $\vec{w}[t]$ is an error term due to unseen distrubances and noise, $u[t]$ is a scalar input, and

$$A = \begin{bmatrix} a_0 & a_1 \\ a_2 & a_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}, \quad \vec{x}[t] = \begin{bmatrix} x_0[t] \\ x_1[t] \end{bmatrix}.$$

To identify the system parameters from measured data, we need to find the unknowns: a_0, a_1, a_2, a_3, b_0 and b_1, however, you can only interact with the system via a blackbox model. The model allows you to view the states $\vec{x}[t] = \begin{bmatrix} x_0[t] & x_1[t] \end{bmatrix}^T$ and it takes a scalar input $u[t]$ that allows the system to move to the next state $\vec{x}[t+1] = \begin{bmatrix} x_0[t+1] & x_1[t+1] \end{bmatrix}^T$.

a) **Write scalar equations for the new states, $x_0[t+1]$ and $x_1[t+1]$ in terms of a_i, b_i, the states $x_0[t]$, $x_1[t]$, and the input $u[t]$.** Here, assume that $\vec{w}[t] = \vec{0}$ (i.e. the model is perfect).

b) Now we want to identify the system parameters. We observe the system at the initial state $\vec{x}[0] = \begin{bmatrix} x_0[0] \\ x_1[0] \end{bmatrix}$, input $u[0]$ and observe the next state $\vec{x}[1] = \begin{bmatrix} x_0[1] \\ x_1[1] \end{bmatrix}$. We can continue this for an m long sequence of inputs.

What is the minimum value of m you need to identify the system parameters?

c) Say we feed in a total of 4 inputs $u[0], u[1], u[2], u[3]$ into our blackbox. This allows us to observe $x_0[0], x_0[1], x_0[2], x_0[3], x_0[4]$ and $x_1[0], x_1[1], x_1[2], x_1[3], x_1[4]$, which we can use to identify the system.

To identify the system we need to set up an approximate (because of potential disturbances) matrix equation

$$D\vec{p} \approx \vec{y}$$

using the observed values above and the unknown parameters we want to find. Suppose you are given the form of D in terms of some of the observed data:

$$D = \begin{bmatrix} x_0[0] & x_1[0] & u[0] & 0 & 0 & 0 \\ x_0[1] & x_1[1] & u[1] & 0 & 0 & 0 \\ x_0[2] & x_1[2] & u[2] & 0 & 0 & 0 \\ x_0[3] & x_1[3] & u[3] & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0[0] & x_1[0] & u[0] \\ 0 & 0 & 0 & x_0[1] & x_1[1] & u[1] \\ 0 & 0 & 0 & x_0[2] & x_1[2] & u[2] \\ 0 & 0 & 0 & x_0[3] & x_1[3] & u[3] \end{bmatrix}.$$

For this D, **what are $\vec{y}$ and the unknowns $\vec{p}$ so that $D\vec{p} \approx \vec{y}$ makes sense?** Tell us what the components of these vectors are, written in vector form.

d) Now that we have set up $D\vec{p} \approx \vec{y}$, **explain how you would use this approximate equation to estimate the unknown values a_0, a_1, a_2, a_3, b_0 and b_1 assuming the columns of D are linearly independent.** In particular, give an expression for your estimate $\widehat{\vec{p}}$ for the unknowns in terms of the D and $\vec{y}$.

(HINT: Don't forget that D is not a square matrix. It is taller than it is wide.)

e) **What could go wrong in the previous case? What kind of inputs would make least-squares fail to give you the parameters you want?**