1 Compact SVD

Recall that an $m \times n$ matrix A will always have a singular value decomposition.

$$A = U\Sigma V^T \tag{1}$$

If A is of rank k, we can write out the SVD in **compact form** as $A = U_c\Sigma_c V_c^T$

$$A = U_c\Sigma_c V_c^T = \sum_{i=1}^{k} \sigma_i \vec{u}_i \vec{v}_i^T \tag{2}$$

where the subscript c is referring to the first k columns of the matrix. U_c is an $m \times k$ matrix, Σ_c is a $k \times k$ matrix and V_c^T is a $k \times n$ matrix.

If $m > n$ and A is a tall matrix of rank n, we can expand the matrices in block form to get the compact SVD where $V = V_c$

$$A = U\Sigma V^T = \begin{bmatrix} U_c & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_c \\ 0 \end{bmatrix} V^T = U_c\Sigma_c V_c^T$$

While if $m < n$, and A is a wide matrix of rank m, then $U_c = U$ and its compact SVD is

$$A = U\Sigma V^T = U \begin{bmatrix} \Sigma_c & 0 \end{bmatrix} \begin{bmatrix} V_c^T \\ V_2^T \end{bmatrix} = U_c\Sigma_c V_c^T$$

Recall that the columns of U_c span Col (A) while the columns of V_2 span Nul (A). Note that the matrix Σ_c is a diagonal matrix with the singular values on its diagonal.

2 Minimum Energy Warmup

Consider an undetermined system of equations given by

$$\vec{y} = A\vec{x}$$

where $A \in \mathbb{R}^{m \times n}$ is a wide matrix, that is $m < n$. We assume that A has linearly independent rows.

a) What is the rank of A?

b) In lecture, we saw that the minimum norm solution for $\vec{x}$ is given by

$$\vec{x} = V_c \Sigma_c^{-1} U_c^T \vec{y} = V_c \Sigma_c^{-1} U^T \vec{y}$$

where the definition of V_c and Σ_c come from the block matrix form of the SVD. That is,

$$A = U \Sigma V^T = U \begin{bmatrix} \Sigma_c & 0_{m \times (n-m)} \end{bmatrix} \begin{bmatrix} V_c^T \\ V_2^T \end{bmatrix} \tag{3}$$

Argue that Σ_c is invertible. What are the matrix elements $\Sigma_{c_{ij}}$ and $\Sigma_{c_{ij}}^{-1}$?

c) Use the SVD of A to show that the expression for the minimum-norm solution from equation 3 can also be written as

$$\vec{x} = A^T (A A^T)^{-1} \vec{y}$$

HINT: $(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$ for invertible matrices $A,\ B,\ C$.

3 Minimum Energy Control

Consider the system

$$\vec{x}(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

Our goal is to reach the target state

$$\vec{x}(5) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

starting at $\vec{x}(0) = 0$.

a) Find the input sequence $u(0), u(1), u(2), u(3), u(4)$ that achieves this with the least possible "energy," as defined by

$$E = u(0)^2 + u(1)^2 + u(2)^2 + u(3)^2 + u(4)^2.$$

Find the value of E for the sequence you computed.

b) Show that, if we reach the target state $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ at $t = 2$ and apply a zero input henceforth ($u(2) = u(3) = u(4) = \cdots = 0$) then

$$\vec{x}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad t = 2, 3, 4, 5, \ldots$$

c) With the result of part (b) in mind, find inputs $u(0), u(1)$ such that

$$\vec{x}(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and apply $u(2) = u(3) = u(4) = 0$ so

$$\vec{x}(5) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

as well. Find the energy E for the resulting input sequence and compare it to the one in part (a).

4 Uncontrollability

Consider the following discrete-time system with the given initial state:

$$\vec{x}(t+1) = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t)$$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

a) Is the system controllable?

b) Is it possible to reach $\vec{x}(T) = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ for some $t = T$? For what input sequence $u(t)$ up to $t = T - 1$?

c) Find the set of all possible states reachable after two timesteps.

d) Is it possible to reach $\vec{x}(T) = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some $t = T$? For what input sequence $u(t)$ up to $t = T - 1$?