1 Continuous and discrete time

There are two different dialects for modeling change over time. Thus far we have modeled real-life events using differential equations and initial conditions. For example, the voltage across a capacitor connected to a voltage source by a resistor is fully described by the following differential equation and initial conditions.

$$\frac{\mathrm{d}}{\mathrm{d}t} v_C(t) = -\frac{1}{RC} v_C(t) + \frac{1}{RC} v_{\mathrm{in}}(t), \quad v_C(0) = v_0 \tag{1}$$

Abstracting away particulars, *continuous-time* scalar linear systems can be represented in variants of the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda x(t) + \mu u(t), \quad x(0) = x_0. \tag{2}$$

This discussion will introduce *discrete-time* scalar linear systems, which have models similar to the following:

$$x[t+1] = ax[t] + bu[t], \quad x[0] = x_0. \tag{3}$$

Notice that evolution is represented by defining the transition from $x[t]$ to $x[t+1]$. The state x is not a continuous function of time, but a sequence of individual moments. Can you think of systems in life that are naturally more susceptible to discrete-time modeling?

2 Dirty Dishes

I am a trip planner who lodges travellers at Bob's Bed and Breakfast. At the beginning of each day, Bob will do half of the dirty dishes in the sink. During the day, each of his guests will use 4 pounds of dishes minus an eighth pound of dishes for each pound of dishes already in the sink at the beginning of the day (as Bob's kitchen gets too messy).

a) What is the state vector for Bob's kitchen sink system? What are the inputs? Write out the state space model.

b) Explain why Bob's kitchen is not a linear system.

c) On Wednesday morning (before Bob gets up), there are 4 pounds of dishes in the sink. On Wednesday, Bob has 4 guests, and on Thursday, he has 5 guests. How many pounds of dishes are in the sink after Thursday?

d) I am a very eccentric trip planner and I want Bob to have exactly 12 pounds of dishes in his sink. He has 24 pounds of dishes in his sink. How many guests should I lodge at Bob's Bed and Breakfast today? How many guests should I lodge tomorrow?

e) Now suppose 5 guests come to Bob's kitchen every day. At the equilibrium state, how many pounds of dishes will remain in the sink?

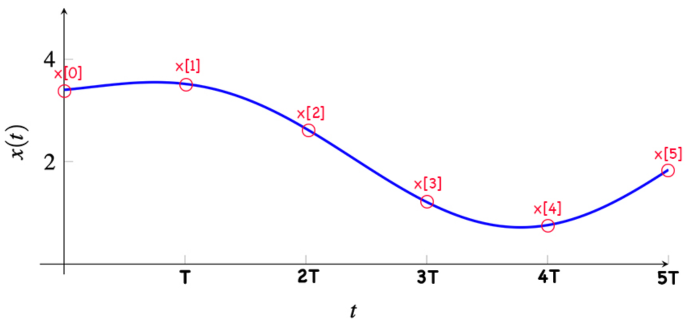
3 Differential equations with piecewise constant inputs

Let $x(\cdot)$ be a solution to the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda \left(x(t) - u(t)\right). \tag{4}$$

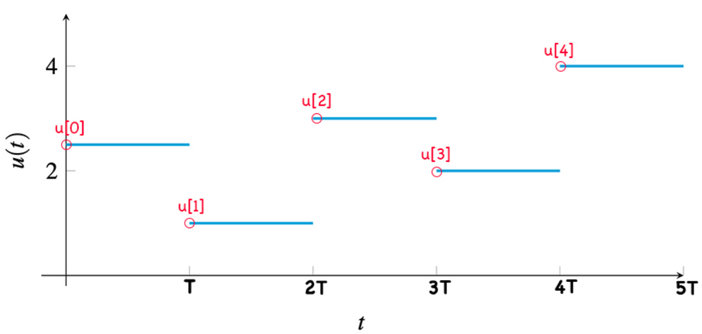
Let $T > 0$. Let $x[\cdot]$ "sample" $x(\cdot)$ as follows:

$$x[n] = x(nT). \tag{5}$$



Assume that $u(\cdot)$ is constant between samples of $x(\cdot)$, i.e.

$$u(t) = u[n] \quad \text{when} \quad nT \leq t < (n+1)T. \tag{6}$$



a) We will approach solving this differential equation iteratively in intervals of size T. What is the solution $x(t)$ for $t \in [0, T)$?

b) Using your solution from the previous part, sample it at $t = T$ to write $x[1]$ in terms of $x[0]$ and $u[0]$.

c) Now for a general time-step n, write $x[n+1]$ in terms of $x[n]$ and $u[n]$. Conclude that the sampled system of a continuous-time linear system is in fact a discrete-time linear system.

d) Let $T = 1$ and $\lambda = -100$. Sketch a piecewise constant input $u[\cdot]$ of your choice, then sketch $x(t)$. Mark $x[n]$. Your sketch doesn't have to be exact, but you should be able to supply analysis to justify why it looks a certain way: how are you using the fact that λT is large and negative?

e) Let $T = 1$ and $\lambda = -1$. Define $u[n]$ as follows:

$$u[n] = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases}. \tag{7}$$

Sketch $x(t)$.