Calculating the Singular Value Decomposition

Suppose we have a matrix A of dimension $m \times n$ where $m > n$ and A has rank r.
We can find the singular value decomposition (SVD)

$$A = U\Sigma V^* = \sum_{i=1}^{n} \sigma_i \vec{u}_i \vec{v}_i^*$$

with the following steps.

(1) Find the eigenvalues λ_i of A^*A and order them such that $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and $\lambda_{r+1} = \cdots = \lambda_n = 0$.

(2) Find the orthonormal eigenvectors of A^*A, so that

$$A^*A\vec{v}_i = \lambda_i \vec{v}_i, \qquad i = 1, \ldots, r$$

Note that the vectors must be orthonormal, that is $\langle \vec{v}_i, \vec{v}_i \rangle = 1$ and $\langle \vec{v}_i, v_j \rangle = 0$ for $i \neq j$.

(3) Let $\sigma_i = \sqrt{\lambda_i}$ and set

$$\vec{u}_i = \frac{A\vec{v}_i}{\sigma_i}, \qquad i = 1, \ldots, r$$

(4) If $r < n$ then we complete the U and V matrices by adding vectors $\vec{u}_{r+1}, \ldots, \vec{v}_m$ and $\vec{v}_{r+1}, \ldots, \vec{v}_n$ to create an orthonormal bases for $\mathbb{R}^m$ and $\mathbb{R}^n$.

1 SVD and Fundamental Subspaces

Define the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}.$$

a) Find the SVD of A.

b) Find the rank of A.

c) Find a basis for the kernel (or nullspace) of A.

d) Find a basis for the range (or columnspace) of A.

e) Repeat parts (a) - (d), but instead, create the SVD of $B = A^*$. What are the relationships between the answers for A and the answers for $B = A^*$?

2 Understanding the SVD

We can compute the SVD for a wide matrix A with dimension m x n where $n > m$ using A^*A with the method described above. However, when doing so you may realize that A^*A is much larger than AA^* for such wide matrices. This makes it more efficient to find the eigenvalues for AA^*. In this question we will explore how to compute the SVD using AA^* instead of A^*A.

a) What are the dimensions of AA^* and A^*A.

b) Given that the $A = U\Sigma V^*$, find a symbolic expression for AA^*.

c) Using the solution to the previous part explain how to find U and Σ from AA^*.

d) Now that we have found the singular values σ_i and the corresponding vectors $\vec{u}_i$ in the matrix U, devise a way to find the corresponding vectors $\vec{v}_i$ in matrix V.

e) Now we have a way to find the vectors $\vec{v}_i$ in matrix V, verify that they are orthonormal.

f) Now that we have found $\vec{v}_i$ you may notice that we only have $m < n$ vectors of dimension n. This is not enough for a basis. How would you complete the m vectors to form an orthonormal basis?

g) Using the previous parts of this question and what you learned from lecture write out a procedure on how to find the SVD for any matrix.