Complex Inner Product

The **complex inner product** $\langle \cdot, \cdot \rangle$ on a vector space V over $\mathbb{C}$ is a function that takes in two vectors and outputs a scalar, such that $\langle \cdot, \cdot \rangle$ is symmetric, linear, and positive-definite.

- Conjugate Symmetry: $\langle \vec{u}, \vec{v} \rangle = \overline{\langle \vec{v}, \vec{u} \rangle}$
- Scaling: $\langle c\vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$ and $\langle \vec{u}, c\vec{v} \rangle = \overline{c}\langle \vec{u}, \vec{v} \rangle$
- Additivity: $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$ and $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
- Positive-definite: $\langle \vec{u}, \vec{u} \rangle \geq 0$ with $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = \vec{0}$

For two vectors, $\vec{u}, \vec{v} \in \mathbb{C}^n$, we usually define their inner product $\langle \vec{u}, \vec{v} \rangle$ to be $\langle \vec{u}, \vec{v} \rangle = \vec{v}^*\vec{u}$. We define the **norm**, or the magnitude, of a vector $\vec{v}$ to be $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{\vec{v}^*\vec{v}}$. For any non-zero vector, we can *normalize*, i.e., set its magnitude to 1 while preserving its direction, by dividing the vector by its norm $\frac{\vec{v}}{\|\vec{v}\|}$.

Adjoint of a Matrix

The **adjoint** or **conjugate-transpose** of a matrix A is the matrix A^* such that $A^*_{ij} = \overline{A_{ji}}$. From the complex inner product, one can show that

$$\langle A\vec{x}, \vec{y} \rangle = \langle \vec{x}, A^*\vec{y} \rangle \tag{1}$$

A matrix is **self-adjoint** or **Hermitian** if $A = A^*$.

Orthogonality and Orthonormality

We know that the angle between two vectors is given by this equation $\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\|\|\vec{v}\| \cos\theta$. Notice that if $\theta = \pm 90°$, the right hand side is 0.

Therefore, we define two vectors $\vec{u}$ and $\vec{v}$ to be **orthogonal** to each other if $\langle \vec{u}, \vec{v} \rangle = \vec{v}^*\vec{u} = 0$. A set of vectors is orthogonal if any two vectors in this set are orthogonal to each other.

Furthermore, we define two vectors $\vec{u}$ and $\vec{v}$ to be **orthonormal** to each other if they are orthogonal to each other and their norms are 1. A set of vectors is orthonormal if any two vectors in this set are orthogonal to each other and every vector has a norm of 1. In fact, for any two vectors $\vec{u}$ and $\vec{v}$ in an orthonormal set,

$$\langle \vec{u}, \vec{v} \rangle = \vec{v}^*\vec{u} = \begin{cases} 1, & \text{if } \vec{u} = \vec{v} \\ 0, & \text{otherwise} \end{cases}.$$

A **unitary** matrix is a square matrix whose columns are orthonormal with respect to the complex inner product.

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}, \qquad \vec{u}_j^*\vec{u}_i = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

Note that $U^*U = UU^* = I$, so the inverse of a unitary matrix is its conjugate transpose $U^{-1} = U^*$.

Since the columns of a unitary matrix are orthonormal vectors, we can interpret these matrices as "rotation" and "reflection" matrices of the standard axes. This also implies that $\|U\vec{v}\| = \|\vec{v}\|$ for any vector $\vec{v}$.

1 Spectral Theorem

For a complex $n \times n$ Hermitian matrix A,

a) All eigenvalues of A are real.

b) A has n linearly independent eigenvectors $\in \mathbb{C}^n$.

c) A has orthogonal eigenvectors, i.e., $A = V\Lambda V^{-1} = V\Lambda V^*$, where Λ is a diagonal matrix and V is a unitary matrix. We say that A is orthogonally diagonalizable.

Recall that a matrix A is Hermitian if $A = A^*$. Furthermore, if A is of the form B^*B for some arbitrary matrix B, all of its eigenvalues are non-negative, i.e., $\lambda \geq 0$.

a) Prove the following: All eigenvalues of a Hermitian matrix A are real.

Hint: Let $(\lambda, \vec{v})$ be an eigenvalue/vector pair and use the definition of an eigenvalue to show that $\lambda\langle\vec{v}, \vec{v}\rangle = \overline{\lambda}\langle\vec{v}, \vec{v}\rangle$.

b) Prove the following: For any Hermitian matrix A, any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

Hint: Use the definition of an eigenvalue to show that $\lambda_1\langle\vec{v}_1, \vec{v}_2\rangle = \lambda_2\langle\vec{v}_1, \vec{v}_2\rangle$.

c) Prove the following: For any matrix A, A^*A is Hermitian and only has non-negative eigenvalues.

2 Fundamental Theorem of Linear Algebra

a) Let $\vec{v}$ be an eigenvector of nonzero eigenvalue of A^*A. Show that $\vec{v} \in \mathrm{Col}(A^*)$.

b) Show that the two subspaces $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^*A)$ are equal.

c) Let $\vec{u}$ be an eigenvector of eigenvalue 0 of A^*A. Show that $\vec{u} \in \mathrm{Nul}(A)$.

d) If A is a $m \times n$ matrix of rank k what are the dimensions of $\text{Col}(A^*)$ and $\text{Nul}(A)$?

e) Use parts (a)-(d) to show that $\text{Col}(A^*)$ is the orthogonal complement of $\text{Nul}(A)$.
Use the spectral theorem on the matrix A^*A to create an orthonormal eigenbasis of $\mathbb{C}^n$