Rational Transfer Functions

When we write the transfer function of an arbitrary circuit, it always takes the following form. This is called a "rational transfer function." We also like to factor the numerator and denominator, so that they become easier to work with and plot:

$$H(\omega) = K\frac{(j\omega)^{N_{z0}}\left(1+j\frac{\omega}{\omega_{z1}}\right)\left(1+j\frac{\omega}{\omega_{z2}}\right)\cdots\left(1+j\frac{\omega}{\omega_{zn}}\right)}{(j\omega)^{N_{p0}}\left(1+j\frac{\omega}{\omega_{p1}}\right)\left(1+j\frac{\omega}{\omega_{p2}}\right)\cdots\left(1+j\frac{\omega}{\omega_{pm}}\right)}$$

Here, we define the constants ω_z as "zeros" and ω_p as "poles."

Bode Plots

Bode plots provide us with a simple and easy tool to plot these transfer functions by hand. Always remember that Bode plots are an approximation; if you want the precisely correct plots, you need to use numerical methods (like solving using MATLAB or IPython).

When we make Bode plots, we plot the frequency and magnitude on a logarithmic scale, and the angle in either degrees or radians. We use the logarithmic scale because it allows us to break up complex transfer functions into its constituent components.

For two transfer functions $H_1(\omega)$ and $H_2(\omega)$, if $H(\omega) = H_1(\omega)\cdot H_2(\omega)$,

$$\log\left|H(\omega)\right| = \log\left|H_1(\omega)\cdot H_2(\omega)\right| = \log\left|H_1(\omega)\right| + \log\left|H_2(\omega)\right| \tag{1}$$

$$\angle H(\omega) = \angle(H_1(\omega)\cdot H_2(\omega)) = \angle H_1(\omega) + \angle H_2(\omega) \tag{2}$$

Decibels

We define the decibel as the following:

$$20\log_{10}(|H(\omega)|) = |H(\omega)|\ [\mathrm{dB}]$$

This means that $20\,\mathrm{dB}$ per decade is equivalent to one order of magnitude. **We won't be using dB when plotting, but understanding the conversion to dB will help when reading the Bode plot sheet on the next page.**

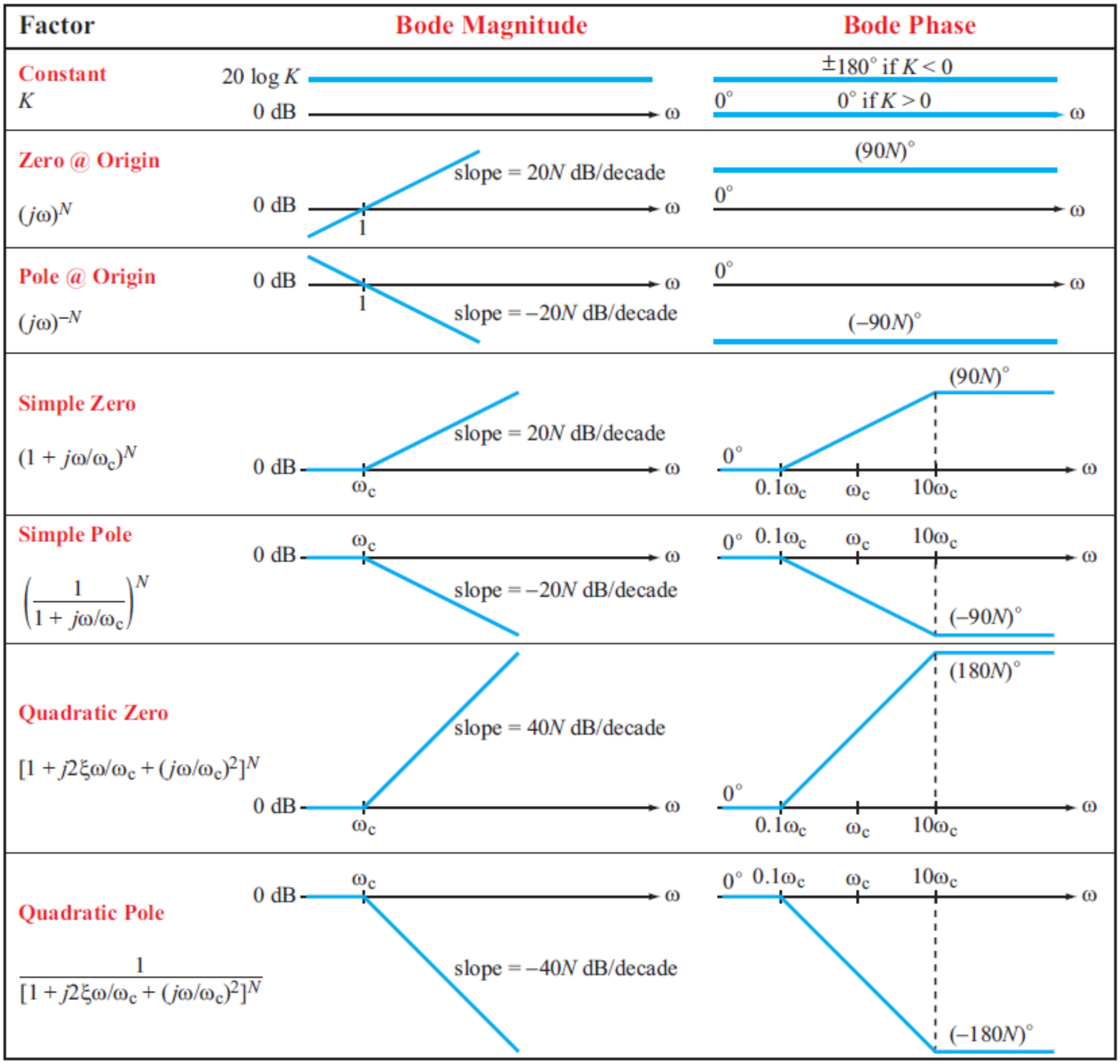
Algorithm

Given a frequency response $H(\omega)$,

a) Break $H(\omega)$ into a product of poles and zeros and put it in "rational transfer function" form. Appropriately divide terms to reduce $H(\omega)$ into one of the given forms. We determine ω_c by reducing a term into one of the above forms.

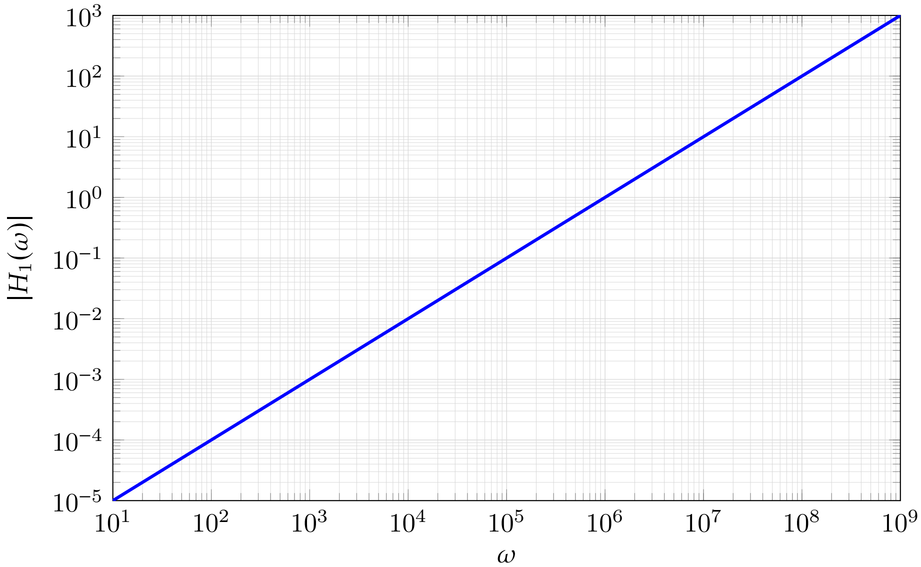
b) Draw out the Bode plot for each pole and zero in the product above.

c) Add the resulting plots to get the final Bode plot.

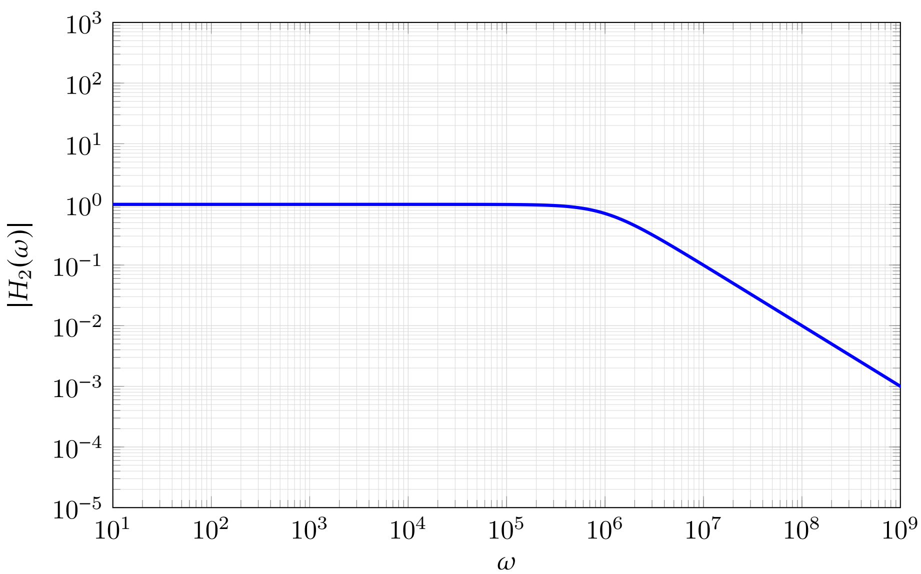

Factor
Bode Magnitude
Bode Phase
Constant
K
20 log K
0 dB
ω
±180° if K < 0
0°
0° if K > 0
ω
Zero @ Origin
(jω)^N
0 dB
1
slope = 20N dB/decade
ω
(90N)°
0°
ω
Pole @ Origin
(jω)^−N
0 dB
1
slope = −20N dB/decade
ω
0°
(−90N)°
ω
Simple Zero
(1 + jω/ωc)^N
0 dB
ωc
slope = 20N dB/decade
ω
(90N)°
0°
0.1ωc
ωc
10ωc
ω
Simple Pole
(1/(1 + jω/ωc))^N
0 dB
ωc
slope = −20N dB/decade
ω
0°
0.1ωc
ωc
10ωc
(−90N)°
ω
Quadratic Zero
[1 + j2ξω/ωc + (jω/ωc)²]^N
0 dB
ωc
slope = 40N dB/decade
ω
(180N)°
0°
0.1ωc
ωc
10ωc
ω
Quadratic Pole
1/[1 + j2ξω/ωc + (jω/ωc)²]^N
0 dB
ωc
slope = −40N dB/decade
ω
0°
0.1ωc
ωc
10ωc
(−180N)°
ω

1 Bode Plot Practice

a) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. What transfer function $H_1(\omega)$ would result in this plot?**

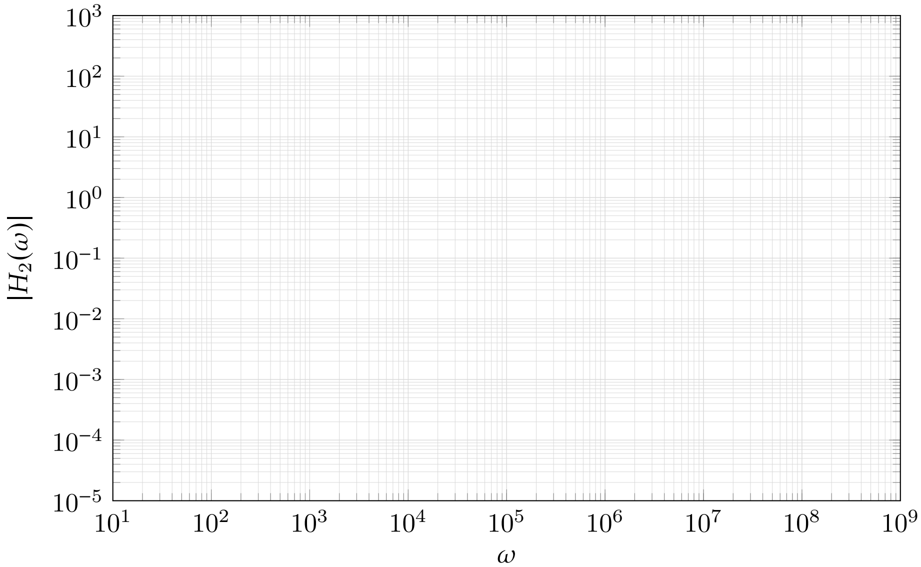


b) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. What transfer function $H_2(\omega)$ would result in this plot?**



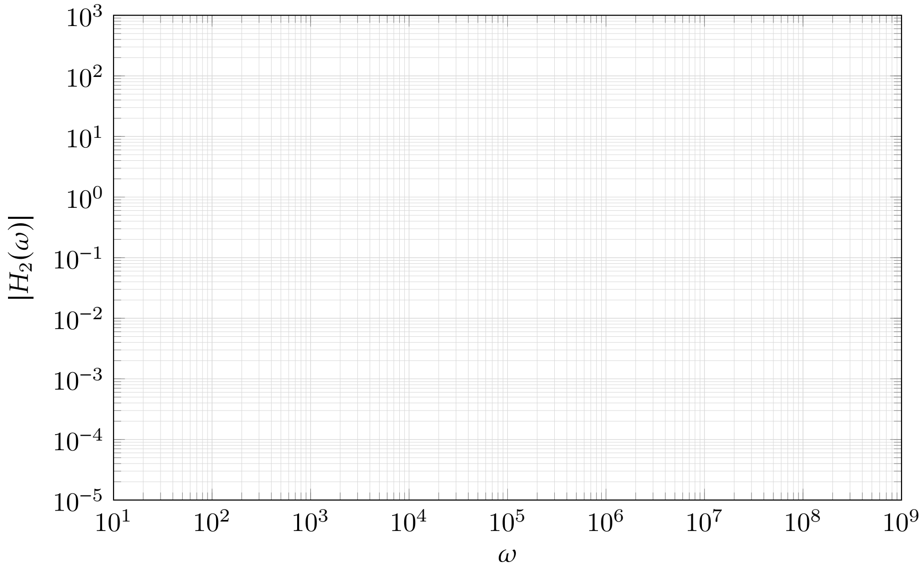
c) **Identify the locations of the poles and zeroes in the following transfer function. Then sketch the magnitude Bode plot.**

$$H_3(\omega) = \frac{\frac{j\omega}{10^6}}{1 + \frac{j\omega}{10^6}}$$



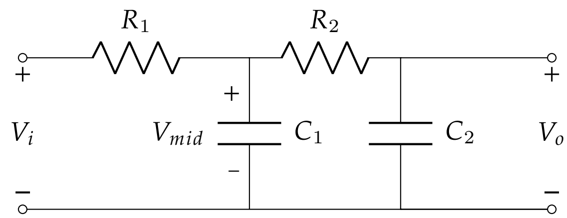
d) **Identify the locations of the poles and zeroes in the following magnitude Bode plot. Then sketch the magnitude Bode plot.**

$$H_4(\omega) = 100\frac{(1+\frac{j\omega}{10^7})^2}{(1+\frac{j\omega}{10^4})(1+\frac{j\omega}{5\times10^4})}$$



2 Transfer functions and why loading is annoying

Consider the circuit below.



The circuit has an input phasor voltage $\widetilde{V}_i$ at frequency ω rad/sec applied at the input terminals shown in the illustration above, causing an output phasor voltage $\widetilde{V}_o$ at output terminals.

a) We are going to construct the transfer function $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i}$ in two steps. We will compute two intermediate transfer functions, $H_1(\omega) = \frac{\widetilde{V}_{mid}}{\widetilde{V}_i}$ and $H_2(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_{mid}}$. Then, we will find the overall transfer function as the product of these two intermediate transfer functions, i.e. $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i} = H_1(\omega)H_2(\omega)$. This approach is valid since the $\widetilde{V}_{mid}$ cancel. For the first step, **find the intermediate transfer function** $H_2(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_{mid}}$**.** Have your expression be in terms of Z_{R2} and Z_{C2}, that is the impedances of R_2 and C_2.

b) Now, **compute the other intermediate transfer function** $H_1(\omega) = \frac{\widetilde{V}_{mid}}{\widetilde{V}_i}$. Have your expression be in terms of Z_{R1}, Z_{R2}, Z_{C1}, and Z_{C2}. (i.e. Don't forget to consider the impact of loading by R_2 and C_2 in this transfer function.)

c) Then, **use these two intermediate transfer functions to calculate the overall transfer function** as $H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i} = H_1(\omega)H_2(\omega)$.

d) Sometimes it is useful to collect all the frequency dependence into one place and to figure out how to think about what scale might be somewhat natural for the frequency. **Obtain an expression for $H(\omega) = \widetilde{V}_o/\widetilde{V}_i$ in the form**

$$H(\omega) = \frac{\widetilde{V}_o}{\widetilde{V}_i} = \frac{1}{1 + 2\xi\frac{j\omega}{\omega_c} + \frac{(j\omega)^2}{\omega_c^2}},$$

given that $R_1 = 2\,\Omega$, $R_2 = 4\,\Omega$, $C_1 = \frac{9}{2}\,\text{F}$, and $C_2 = 1\,\text{F}$. What are the values of ξ and ω_c?

e) **For the previous case, what is the magnitude of the transfer function at the $\omega = \omega_c$ you calculated?**

This is here so that you can see that just because we called it ω_c doesn't mean that the amplitude here is $\frac{1}{\sqrt{2}}$.

f) We can express the transfer function $H(\omega)$ in the polar form. That is,

$$H(\omega) = M(\omega)e^{j\phi(\omega)}$$

The functions $M(\omega)$ and $\phi(\omega)$ are the magnitude and the phase angle of $H(\omega)$, respectively. **Write down $M(\omega)$ and $\phi(\omega)$ using the transfer function you derived in part (b).**

g) Use a computer and then **draw Bode Plots of $|H(\omega)|$ and $\angle H(\omega)$.**