Circulant Matrices

A square matrix C_h is circulant if each row vector is rotated one element to the right relative to the preceding row vector.

$$C_h = \begin{bmatrix} h_0 & h_{N-1} & \cdots & h_2 & h_1 \\ h_1 & h_0 & h_{N-1} & & h_2 \\ \vdots & h_1 & h_0 & \ddots & \vdots \\ h_{N-2} & \vdots & \ddots & \ddots & h_{N-1} \\ h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{bmatrix} \tag{1}$$

Recall from lecture that we can describe the input-output relationship of a periodic discrete-time LTI system via a circulant matrix.

$$\vec{y} = C_h \vec{x} \tag{2}$$

In this case, the first column of C_h is the impulse response $h[n]$ of the system.

$$\vec{h} = \begin{bmatrix} h_0 & h_1 & \cdots & h_{N-2} & h_{N-1} \end{bmatrix} \tag{3}$$

Rather beautifully, the DFT basis vectors are eigenvectors of C_h. We will have N DFT vectors, since that is the dimensionality of our model.

$$\vec{u_k} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & e^{j\frac{2\pi}{N}k\cdot 1} & \cdots & e^{j\frac{2\pi}{N}k\cdot(N-1)} \end{bmatrix} \tag{4}$$

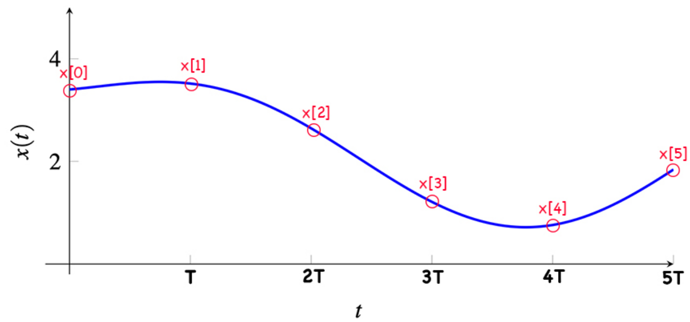
Letting $H[k]$ be the k^{th} DFT coefficient of $h[n]$, we can write the following eigenvalue equation for $k = 0, 1, \cdots, N-1$.

$$C_h \vec{u}_k = \underbrace{\left(\sqrt{N} \times H[k]\right)}_{\text{eigenvalue}} \vec{u}_k \tag{5}$$

In this discussion you'll see why this is useful by representing convolution as a circulant matrix C_h, and then diagonalizing it. This will draw the connection between the DFT and LTI systems.

Sampling theorem

Let x be continuous signal bandlimited by frequency ω_{max}. We sample x with a period of T_s.



Given the discrete samples, we can try reconstructing the original signal f through sinc-interpolation where $\Phi(t) = \text{sinc}\left(\frac{t}{T_s}\right)$

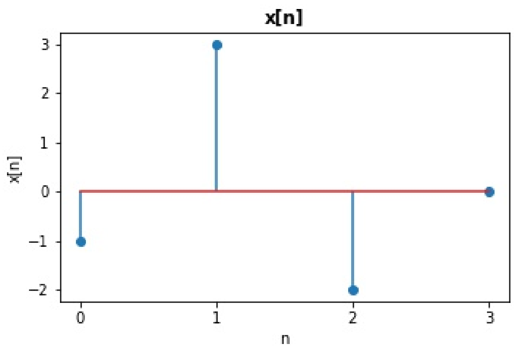
$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x[n]\Phi(t - nT_s)$$

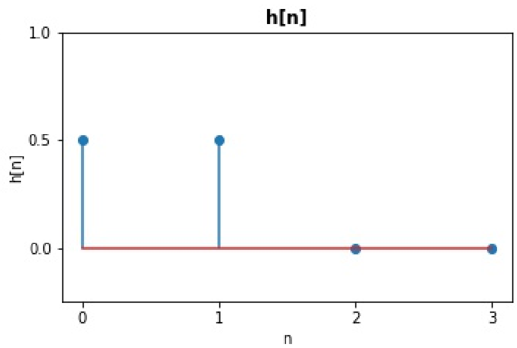
We define the **sampling frequency** as $\omega_s = \frac{2\pi}{T_s}$. The Sampling Theorem says if $\omega_{max} < \frac{\pi}{T_s}$, or $\omega_s > 2\omega_{max}$, then we are able to recover the original signal, i.e. $x = \hat{x}$.

1 Circulant Matrices & Convolution

Consider the signal $x[n]$ of length 3 and an impulse response $h[n]$ of length 2. You may assume that they are zero everywhere else.

$$\vec{x} = \begin{bmatrix} -1 & 3 & -2 \end{bmatrix}^T \qquad \vec{h} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}^T \tag{6}$$





a) What is the convolution $y[n] = x[n] * h[n]$? Also what is the length of this output signal?

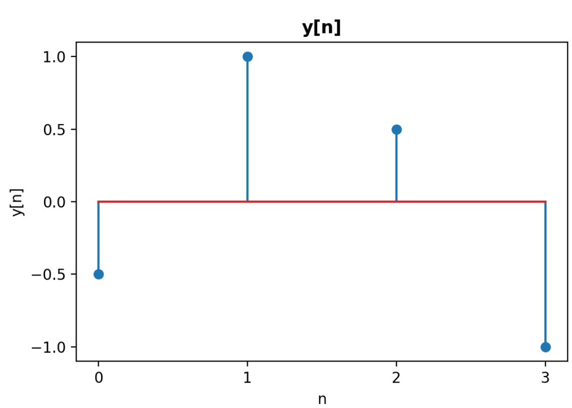
Answer

We can find the convolution by writing out the summation formula and the nonzero terms will remain

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=0}^{\infty} x[k]h[n-k] \tag{7}$$

$$\begin{aligned} y[0] &= x[0]h[0] = -0.5 \\ y[1] &= x[0]h[1] + x[1]h[0] = 1 \\ y[2] &= x[1]h[1] + x[2]h[0] = 0.5 \\ y[3] &= x[2]h[1] = -1 \end{aligned}$$

The length of the output is 4 and we show a visual of the result below



b) Now write each term of the output signal $y[n]$ as a sum using the convolution formula and set up a matrix equation $\vec{y} = A\vec{x}$. What is the size of this matrix?

Answer

$$\begin{aligned} y[0] &= x[0]h[0] = -0.5 \\ y[1] &= x[0]h[1] + x[1]h[0] = 1 \\ y[2] &= x[1]h[1] + x[2]h[0] = 0.5 \\ y[3] &= x[2]h[1] = -1 \end{aligned}$$

We can write this as the following matrix-vector equation

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 \\ h[1] & h[0] & 0 \\ 0 & h[1] & h[0] \\ 0 & 0 & h[1] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \end{bmatrix}$$

The matrix A is 4×3.

c) Add elements to the matrix A and zeros to the vector $\vec{x}$ to create a square matrix C_h that is circulant.

Answer

Note the first three rows of the matrix follow the pattern of a circulant matrix. Therefore, we will add one more cycle as columns and pad a zero to $\vec{x}$ to get

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & h[1] \\ h[1] & h[0] & 0 & 0 \\ 0 & h[1] & h[0] & 0 \\ 0 & 0 & h[1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ 0 \end{bmatrix}$$

d) Since the DFT diagonalizes circulant matrices, lets try to solve for the output signal $y[n]$ using the DFT instead of convolution.

- Step 1: Compute the DFT of $x[n]$ and $h[n]$: $\vec{X} = F\vec{x}$, $\vec{H} = F\vec{h}$.
- Step 2: Take the elementwise product of the DFTs and scale: $\vec{Y} = \sqrt{N}\vec{X} \odot \vec{H}$.
- Step 3: Perform the inverse DFT to get the result $\vec{y} = F^*\vec{Y}$.

Answer

Since $N = 4$, the DFT and IDFT matrices are as follows

$$F = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & 1 & -j \end{bmatrix} \qquad F^* = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & 1 & j \end{bmatrix}$$

- Step 1: Compute the DFT of both signals $x[n]$ and $h[n]$

$$\vec{X} = F\vec{x} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 - 1.5j \\ -3 \\ 0.5 + 1.5j \end{bmatrix}$$

$$\vec{H} = F\vec{h} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.25 - 0.25j \\ 0 \\ 0.25 + 0.25j \end{bmatrix}$$

- Step 2: Take the elementwise product of the DFTs and scale $\vec{Y} = \sqrt{N}\vec{X} \odot \vec{H}$.

$$Y[k] = 2 \cdot X[k] \odot H[k] = \begin{bmatrix} 0 \\ -0.5 - j \\ 0 \\ -0.5 + j \end{bmatrix}$$

- Step 3: Perform the inverse DFT to get the result $\vec{y} = F^* \vec{Y}$.

$$y[n] = F^* Y[k] = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & = 1 & j \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 - j \\ 0 \\ -0.5 + j \end{bmatrix} = \begin{bmatrix} -0.5 \\ 1 \\ 0.5 \\ -1 \end{bmatrix}$$

e) What is the importance behind this result? Compare the runtimes between convolution and the Fast Fourier Transform (FFT) which takes $O(N \log N)$ operations.

Answer

If $x[n]$ and $h[n]$ are signals of length N, then convolution as matrix-vector multiplication takes $O(N^2)$ operations. On the other hand, the DFT can be computed using $O(N \log N)$ operations through the FFT. This means we can find the output of any LTI system efficiently using the FFT.

As a reference for $N = 10^6$, convolution will take approximately 1 trillion operations while the FFT takes approximately 6 million operations. When ran in numpy for $N = 10^6$, convolution took 20 minutes while the FFT took 0.25 seconds.

2 Sampling Theorem basics

Consider the following signal, $x(t)$ defined as,

$$x(t) = \cos(2\pi t)$$

a) Find the maximum frequency, $\omega_{\max}$, in radians per second? In Hertz? (From now on, frequencies will refer to radians per second.)

Answer

$\omega_{\max} = 2\pi$ in radians per second, which is 1 Hertz.

b) If I sample every T seconds, what is the sampling frequency?

Answer

$\omega_s = \frac{2\pi}{T}$.

c) What is the smallest sampling period T that would result in an imperfect reconstruction?

Answer

From the sampling theorem, we know that T has an upperbound of $\frac{\pi}{\omega_{max}}$ for perfect reconstruction. Hence the smallest T for which we cannot reconstruct our signal is,

$$T = \frac{\pi}{2\pi} = \frac{1}{2}$$

.

3 More Sampling

Let's sample the signal from the previous question x with sampling period $T_m = \frac{1}{4}$s and $T_n = 1$s and perform sinc interpolation on the resulting samples. Let the reconstructed functions be f_m and f_n.

a) Have we satisfied the Nyquist limit (i.e. the sampling theorem) in any case?

Answer

To satisfy the Nyquist limit, we need the sampling period $T < \frac{1}{2}$. Hence, T_m satisfies Nyquist, but T_n does not.

b) What is the highest frequency we can reconstruct with the sampling rate T_n?

Answer

The sinc functions used to reconstruct f_n are,

$$\left\{\operatorname{sinc}\left(\frac{t-k}{1}\right)\right\}_{k\in\mathbb{Z}}.$$

These functions can represent a maximum frequency of π.

c) Based on this answer, can you think of any periodic function that has a frequencies less than or equal to π that samples the same as f_n?

Answer

Since the frequencies vary from 0 to π, the smallest period that can be represented is 2. That is to say, functions of period < 2 cannot be captured with the sinc function derived from T_n. Since the period must be greater than 2, no sine or cosin function can give the same samples as f_n. This means suggests looking into a fairly trivial kind of periodic function: a constant. In particular, the answer to this problem is the constant function that is 1 everywhere.

4 Aliasing

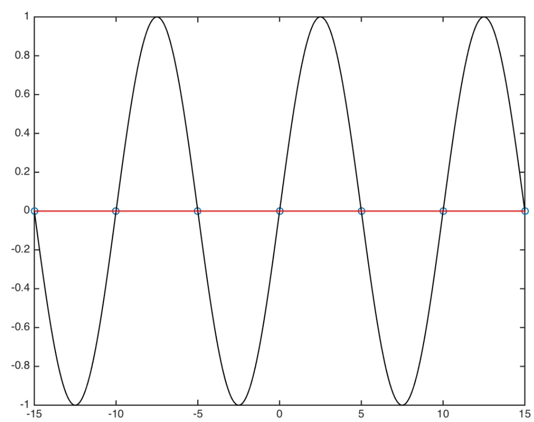
Consider the signal $x(t) = \sin(0.2\pi t)$.

a) At what period T should we sample so that sinc interpolation recovers a function that is identically zero?

Answer

We want to sample such that our resultant discrete time signal is all zeros. To do this, we can sample at $t = 5k$, for integral values of k. Hence, T = 5.

We could also do this graphically by plotting $x(t) = \sin(0.2\pi t)$ and $x(t) = 0$ on the same plot and seeing where they interesect.



b) At what period T can we sample at so that sinc interpolation recovers the function $f(t) = -\sin\left(\frac{\pi}{15}t\right)$?

Answer

T = 7.5

$$
\begin{aligned}
x[n] &= \sin(0.2\pi nT) && \text{sampling x(t)}\\
&= -\sin(-0.2\pi nT) && \sin(t) \text{ is odd}\\
&= -\sin(-0.2\pi nT + 2\pi n) && \text{For } n \in \mathbb{Z} \text{ since } \sin(t) \text{ is periodic.}\\
&= -\sin\left(\frac{\pi}{15}nT\right)
\end{aligned}
$$

As a result,

$$
\begin{aligned}
2\pi - 0.2\pi T &= \frac{\pi}{15} T \\
T &= 7.5
\end{aligned}
$$

As with part (a), we could also do this graphically by plotting $x(t) = \sin(0.2\pi t)$ and $x(t) = -\sin\left(\frac{\pi}{15} t\right)$ on the same plot and looking at the intersection points.