1 Discrete Time Systems

Consider a discrete-time system with $x[n]$ as input and $y[n]$ as output.

$$x[n] \longrightarrow \boxed{} \longrightarrow y[n]$$

The following are some of the possible properties that a system can have:

Linearity

A **linear system** has the properties below:

1. **additivity**

$$x_1[n] + x_2[n] \longrightarrow \boxed{} \longrightarrow y_1[n] + y_2[n] \tag{1}$$

2. **scaling**

$$\alpha x[n] \longrightarrow \boxed{} \longrightarrow \alpha y[n] \tag{2}$$

Here, α is some constant.
Together, these two properties are known as **superposition**:

$$\alpha_1 x_1[n] + \alpha_2 x_2[n] \longrightarrow \boxed{} \longrightarrow \alpha_1 y_1[n] + \alpha_2 y_2[n]$$

Time Invariance

A system is **time-invariant** if its behavior is fixed over time:

$$x[n - n_0] \longrightarrow \boxed{} \longrightarrow y[n - n_0] \tag{3}$$

Causality

A **causal** system has the property that $y[n_0]$ only depends on $x[n]$ for $n \in (-\infty, n_0]$. An intuitive way of interpreting this condition is that the system does not "look ahead."

Bounded-Input, Bounded-Output (BIBO) Stability

In a BIBO stable system, if $x[n]$ is bounded, then $y[n]$ is also bounded. A signal $x[n]$ is bounded if there exists an M such that $|x[n]| \leq M < \infty \; \forall n$.

2 Linear Time-Invariant (LTI) Systems

A system is LTI if it is both linear and time-invariant. We define the **impulse response** of an LTI system as the output $h[n]$ when the input $x[n] = \delta[n]$ where $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$.

An LTI system can be uniquely characterized by its impulse response $h[n]$. In addition, the following properties hold:

- An LTI system is causal iff $h[n] = 0 \ \forall n < 0$.
- An LTI system is BIBO stable iff its impulse reponse is absolutely summable:

$$\sum_{n=-\infty}^{\infty} \left|h[n]\right| < \infty$$

Convolution Sum

Consider the following LTI system with impulse reponse $h[n]$:

$$x[n] \longrightarrow \boxed{} \longrightarrow y[n]$$

Notice that we can write $x[n]$ as a sum of impulses:

$$x[n] = \ldots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \ldots = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

In addition, we know that:

$$\delta[n] \longrightarrow \boxed{} \longrightarrow h[n]$$

By applying the LTI property of our system, we get that

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

The expression $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$ is referred to as the **convolution sum** and can be written as $x[n] * h[n]$ or $(x * h)[n]$.

3 Is it LTI?

Determine if the following systems are LTI:

a) $y[n] = 4x[n]$

Answer

LTI.

b) $y[n] = 2x[n] - 4$

Answer

Not linear. Let $\hat{x}[n] = 2x[n]$. Then $\hat{y}[n] = 4x[n] - 4 \neq 2y[n]$.

c) $y[n] = 2x[-2+3n] + 2x[2+3n]$

Answer

Linear, not time-invariant.

Let $\hat{x}[n] = x[n-n_0]$ be a delayed input signal. Then, the corresponding output $\hat{y}[n]$ is equal to $2x[-2+3n-n_0] + 2x[2+3n-n_0]$.

However, we can see that $\hat{y}[n] \neq y[n-n_0] = 2x[-2+3(n-n_0)] + 2x[2+3(n-n_0)]$.

d) $y[n] = 4^{x[n]}$

Answer

Non-linear.

Let $\hat{x}[n] = 2x[n]$. Then $\hat{y}[n] = 16^{x[n]} \neq 2y[n]$.

e) $y[n] - y[n-1] = x[n]$

Answer

(i) **Linearity:**

- Scaling:
  Let $x[n]$ be an input with output $y[n]$. Then if we input $\hat{x}[n] = \alpha x[n]$,

$$\hat{x}[n] = \alpha x[n] = \alpha(y[n] - y[n-1]) = \alpha y[n] - \alpha y[n-1]$$

  This implies that $\hat{y}[n] = \alpha y[n]$.

- Additivity:
  Let $x_1[n]$ and $x_2[n]$ be inputs with outputs $y_1[n]$ and $y_2[n]$. Then if we input $\hat{x}[n] = (x_1 + x_2)[n]$,

$$\begin{aligned}\hat{x}[n] = x_1[n] + x_2[n] &= y_1[n] - y_1[n-1] + y_2[n] - y_2[n-1] \\ &= y_1[n] + y_2[n] - y_1[n-1] - y_2[n-1]\end{aligned}$$

  This shows that $\hat{y}[n] = y_1[n] + y_2[n]$ is the output.

(ii) **Time-Invariance**

Let $\hat{x}[n] = x[n-n_0]$ be a delayed input signal. We see that

$$\hat{x}[n] = x[n-n_0] = y[n-n_0] - y[n-n_0-1]$$

As a result, the output $\hat{y}[n]$ must be $\hat{y}[n] = y[n-n_0]$.

We conclude by saying the system is LTI.

f) $y[n] = x[n] + nx[n-1]$

Answer

Not time-invariant.

4 Convoluted Convolution

a) Show that convolution is commutative. That is, show that $(x * h)[n] = (h * x)[n]$.

Answer

$$\begin{aligned}(x * h)[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{m=-\infty}^{\infty} x[n-m]h[m] && \text{Let } m = n-k. \\ &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] \\ &= (h * x)[n]\end{aligned}$$

b) Show that $\delta[n]$ is a convolution identity. That is, show that $(x * \delta)[n] = x[n]$.

Answer

Since convolution is commutative, we know that $(x * \delta)[n] = (\delta * x)[n]$.

$$(\delta * x)[n] = \sum_{k=-\infty}^{\infty} \delta[k]x[n-k]$$

Since $\delta[k] = 0$ for all $k \neq 0$, it follows that

$$(\delta * x)[n] = \delta[0]x[n] = x[n]$$

c) Show that convolution by $\delta[n - n_0]$ shifts $x[n]$ by n_0 steps to the right.

Answer

Since convolution is commutative $x[n] * \delta[n-n_0] = \delta[n-n_0] * x[n]$.

$$\delta[n-n_0] * x[n] = \sum_{k=-\infty}^{\infty} \delta[k-n_0]x[n-k]$$

Then since $\delta[k-n_0] = 0$ for all $k \neq n_0$, it follows that

$$\delta[n-n_0] * x[n] = \delta[0]x[n-n_0] = x[n-n_0]$$

d) Show that convolution is distributive. In other words, show that $(x * (h_1 + h_2))[n] = (x * h_1)[n] + (x * h_2)[n]$.

Answer

Since multiplication is distributive, it follows that convolution is distributive

$$\begin{aligned}(x * (h_1 + h_2))[n] &= \sum_{k=-\infty}^{\infty} x[k](h_1[n-k] + h_2[n-k]) \\ &= \sum_{k=-\infty}^{\infty} x[k]h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k]h_2[n-k] \\ &= (x * h_1)[n] + (x * h_2)[n]\end{aligned}$$

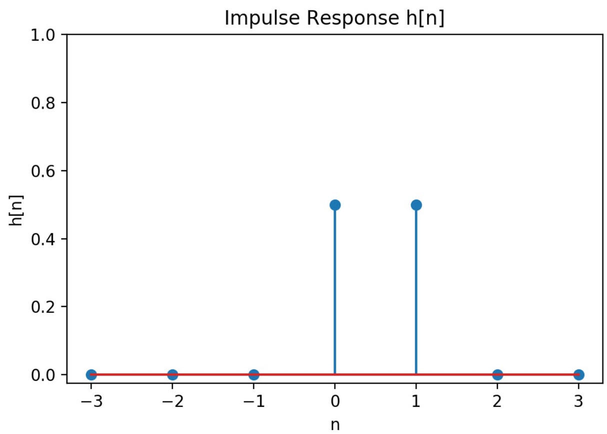
5 Mystery System

Consider an LTI system with the following impulse response:

$$h[n] = \frac{1}{2}(\delta[n] + \delta[n-1])$$

a) Create a sketch of this impulse reponse.

Answer



b) What is the output of our system if the input is the unit step $u[n]$?

Answer

$$y[n] = (u * h)[n] = \sum_{k=-\infty}^{\infty} u[k]h[n-k] = \sum_{k=0}^{\infty} h[n-k]$$

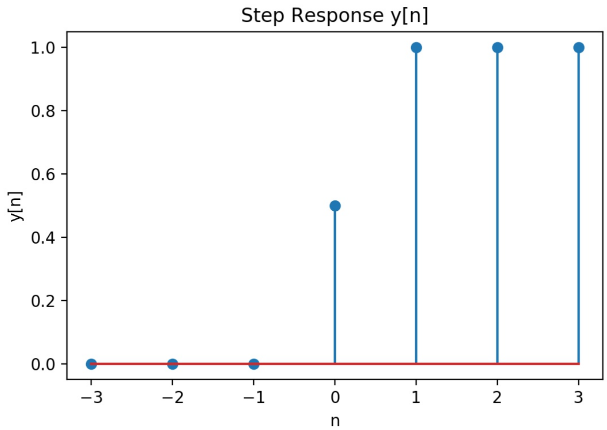
For $n < 0$, $y[n] = 0$. When $n > 0$,

$$y[0] = \sum_{k=0}^{\infty} h[-k] = h[0] = 0.5$$

$$y[1] = \sum_{k=0}^{\infty} h[1-k] = h[0] + h[1] = 1$$

$$y[n] = \sum_{k=0}^{\infty} h[n-k] = h[0] + h[1] + \ldots + h[n] = 1 \text{ for } n > 1.$$

The output $y[n]$ is shown below.



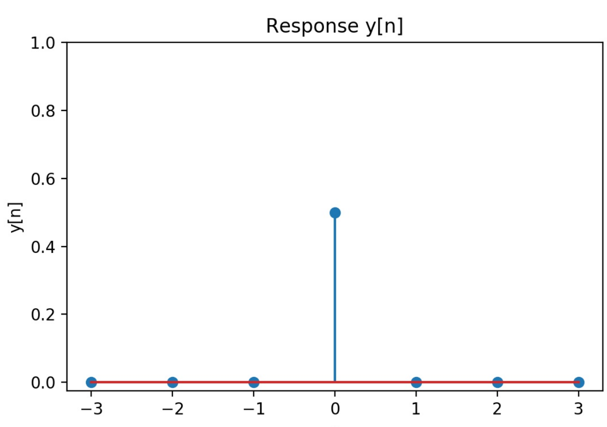
c) What is the output of our system if our input is $x[n] = (-1)^n u[n]$?

Answer

$$y[n] = (u * h)[n] = \sum_{k=-\infty}^{\infty} x[n]h[n-k] = \sum_{k=0}^{\infty}(-1)^k h[n-k]$$

For $n < 0$, $y[n] = 0$. When $n > 0$,

$$\begin{aligned}
y[0] &= \sum_{k=0}^{\infty} h[-k] = h[0] = 0.5 \\
y[1] &= \sum_{k=0}^{\infty} h[1-k] = h[0] - h[1] = 0 \\
y[2] &= \sum_{k=0}^{\infty} h[2-k] = h[2] - h[1] + h[0] = 0 \\
&\vdots \\
y[n] &= 0 \text{ for } n > 0.
\end{aligned}$$



d) This system is called the two-point simple moving average (SMA) filter. Based on the previous parts, why do you think it bears this name?

Answer

The output of the system at each timestep n is the average of $x[n]$ and $x[n-1]$. To show this formally, we can look at the convolution $y = x * h$

$$\begin{aligned} y[n] = (x * h)[n] &= x * (\frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]) \\ &= \frac{1}{2}x[n] + \frac{1}{2}x[n-1] \end{aligned}$$

This sort of system can be used in areas like technical analysis to gain insight into stock prices and trends (usually these methods would use a longer window than just two days). There are also other variants used like the exponential moving average (EMA) filter.