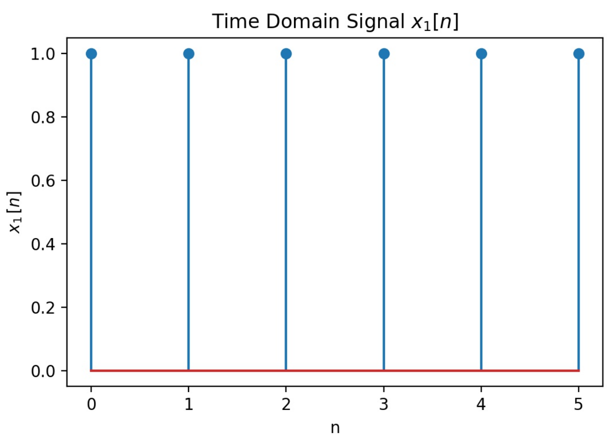
1 DFT

Consider the following length 6 signals. Compute its DFT coefficients $X[k]$. Then plot its magnitude $|X[k]|$ and phase $\angle X[k]$.

a) $x_1[n] = u[n] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

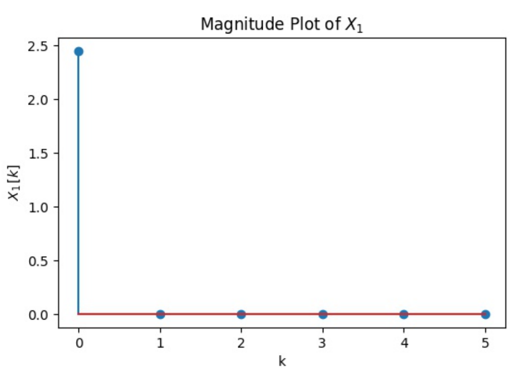


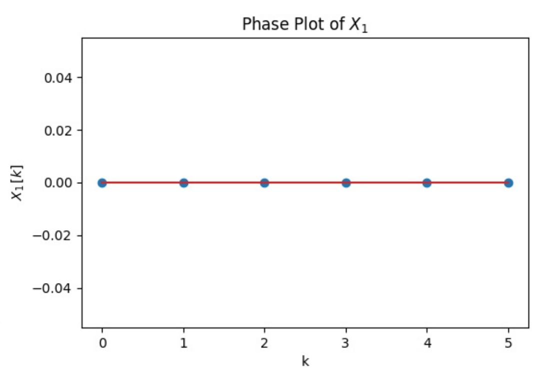
Answer

Since $x_1[n] = \sqrt{6} u_0[n]$ where u_0 is the DC component DFT basis vector, the frequency components must be

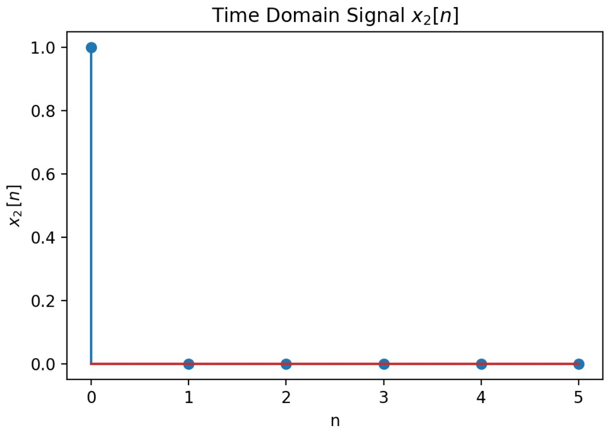
$$X_1[k] = \begin{cases} \sqrt{6} & k = 0. \\ 0 & k \neq 0. \end{cases}$$

The magnitude and phase of X_1 as plotted below





b) $x_2[n] = \delta[n] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

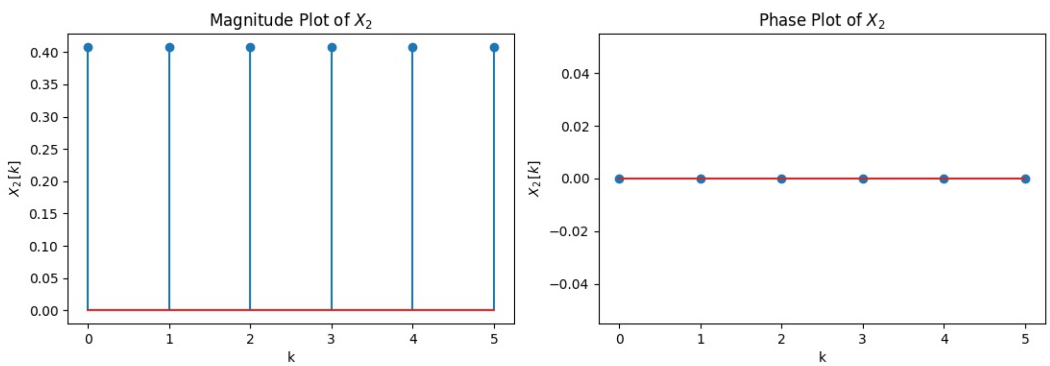


Answer

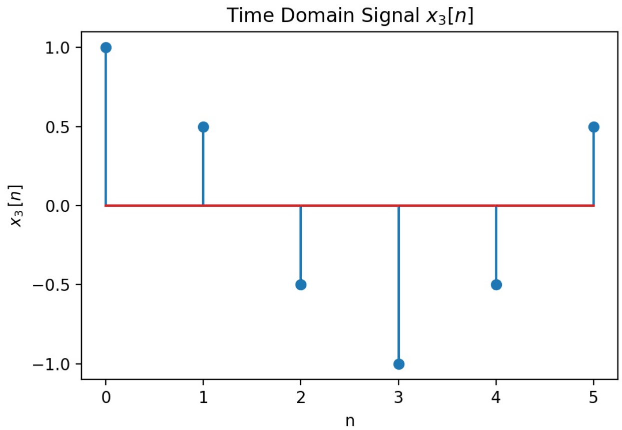
We can compute the frequency components by multiplying by the matrix $F = U^*$. Since $\delta[n]$ is zero for $n > 0$, the frequency components will be the first column of F.

$$X_2[k] = Fx_2[n] = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

The magnitude and phase of X_2 as plotted below



c) $x_3[n] = \cos\left(\frac{2\pi}{6}n\right)$ for $n = 0, 1, \ldots, 5$.



Answer

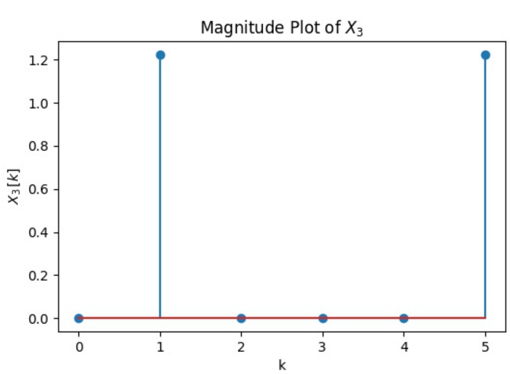
$$\cos\left(\frac{2\pi}{6}n\right) = \frac{1}{2}e^{j\frac{2\pi}{6}n} + \frac{1}{2}e^{-j\frac{2\pi}{6}n}$$

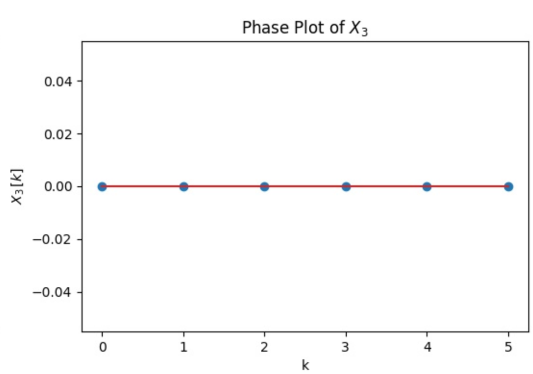
$$u_k[n] = \frac{1}{\sqrt{6}}e^{j\frac{2\pi}{6}kn}$$

$$\vec{x}_3 = \frac{\sqrt{6}}{2}(\vec{u}_1 + \vec{u}_5)$$

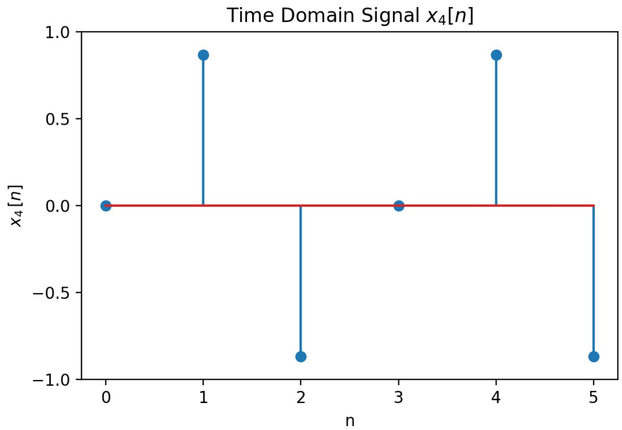
$$X_3[k] = \begin{cases} \frac{\sqrt{6}}{2} & k = 1, 5. \\ 0 & k \neq 1, 5. \end{cases}$$

The magnitude and phase of X_3 as plotted below





d) $x_4[n] = \sin\left(\frac{4\pi}{6}n\right)$ for $n = 0, 1, \ldots, 5$.

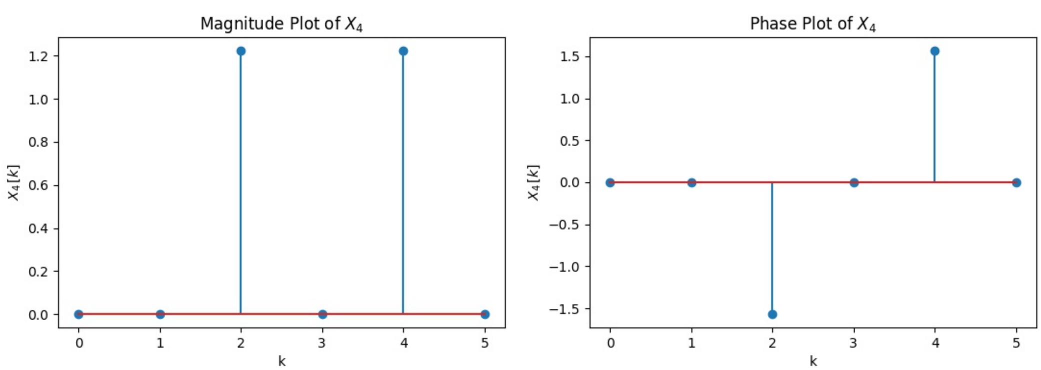


Answer

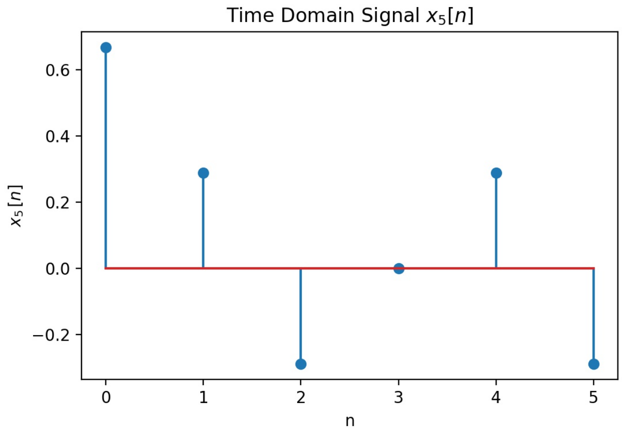
$$\sin\left(\frac{4\pi}{6}n\right) = \frac{1}{2j}e^{j\frac{4\pi}{6}n} - \frac{1}{2j}e^{-j\frac{4\pi}{6}n}$$
$$u_k[n] = \frac{1}{\sqrt{6}}e^{j\frac{2\pi}{6}kn}$$
$$\vec{x}_4 = \frac{\sqrt{6}}{2j}(\vec{u}_2 - \vec{u}_4)$$

$$X_4[k] = \begin{cases} \frac{-\sqrt{6}j}{2} & k = 2 \\ \frac{\sqrt{6}j}{2} & k = 4 \\ 0 & k \neq 2, 4. \end{cases}$$

The magnitude and phase of X_4 as plotted below



e) $x_5[n] = \frac{2}{3}x_2[n] + \frac{1}{3}x_4[n]$

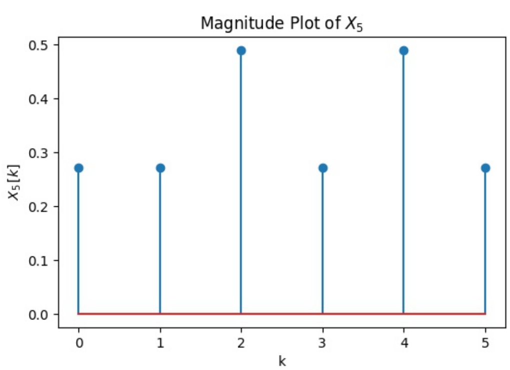


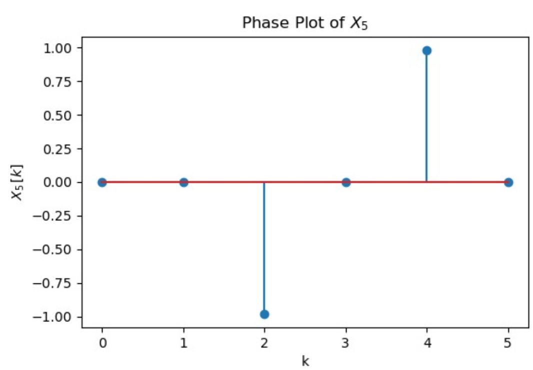
Answer

Since the DFT is linear, $X_5[k] = \frac{2}{3}X_2[k] + \frac{1}{3}x_4 X_4[k]$. Therefore, we can write out the DFT coefficients of x_5 as

$$\begin{aligned} X_5[k] &= \frac{2}{3}\begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{bmatrix} + \frac{1}{3}\begin{bmatrix}0 & 0 & \frac{-\sqrt{6}j}{2} & 0 & \frac{\sqrt{6}j}{2} & 0\end{bmatrix} \\ &= \begin{bmatrix}\frac{2}{3\sqrt{6}} & \frac{2}{3\sqrt{6}} & \frac{2}{3\sqrt{6}} - \frac{\sqrt{6}j}{6} & \frac{2}{3\sqrt{6}} & \frac{2}{3\sqrt{6}} + \frac{\sqrt{6}j}{6} & \frac{2}{3\sqrt{6}}\end{bmatrix} \end{aligned}$$

The magnitude and phase of X_5 as plotted below

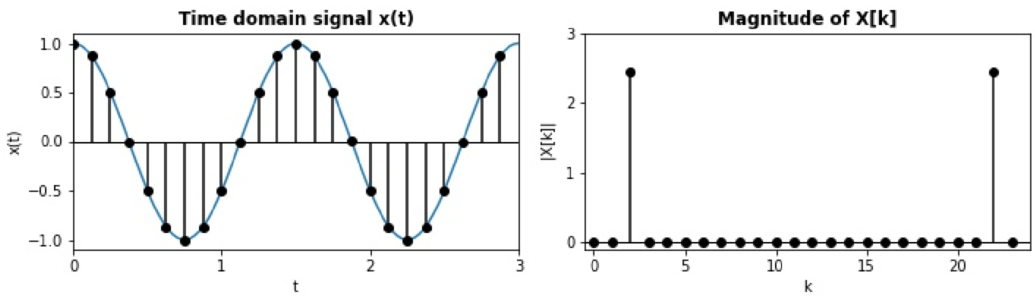




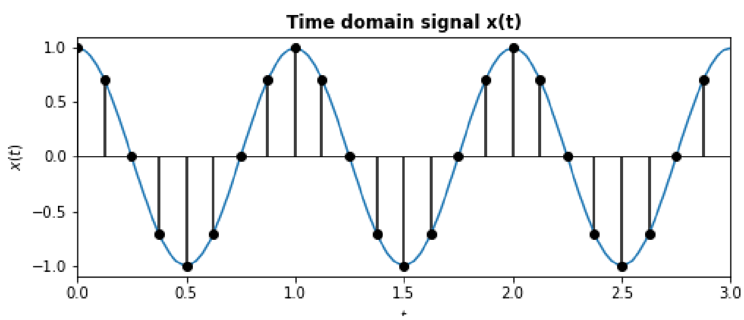
2 DFT Sampling Matching

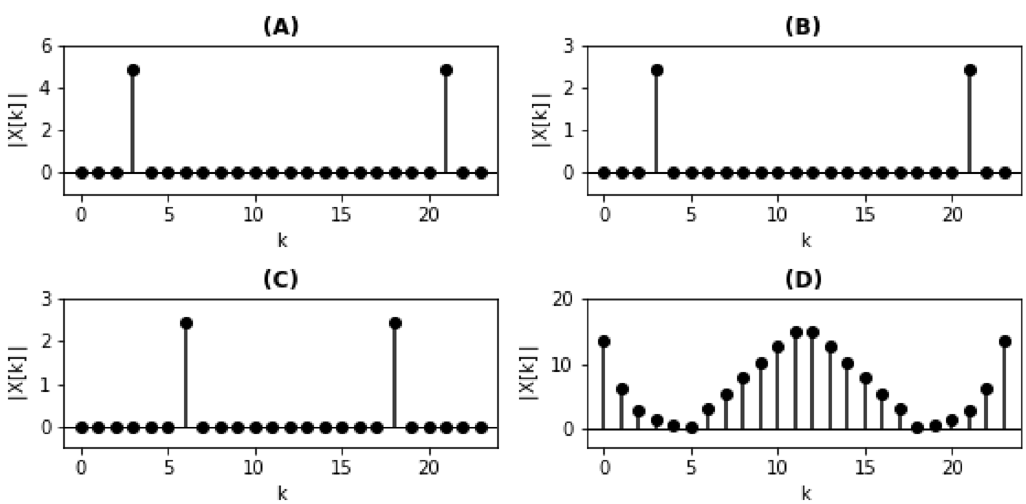
Select the correct answer from the multiple choice options provided and give some justification.

a) A sampled time domain signal and its DFT coefficients are given below:



Now given the following time domain signal, which of the options below shows the correct DFT coefficient magnitudes?

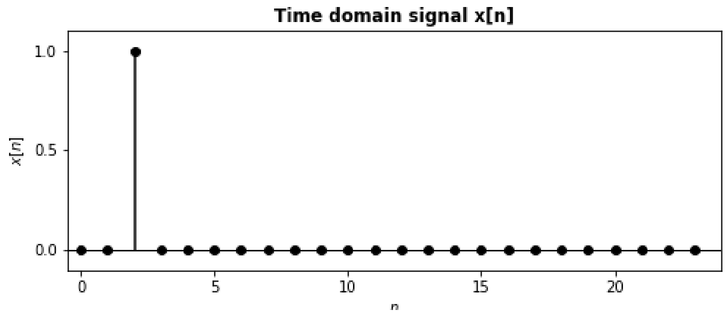


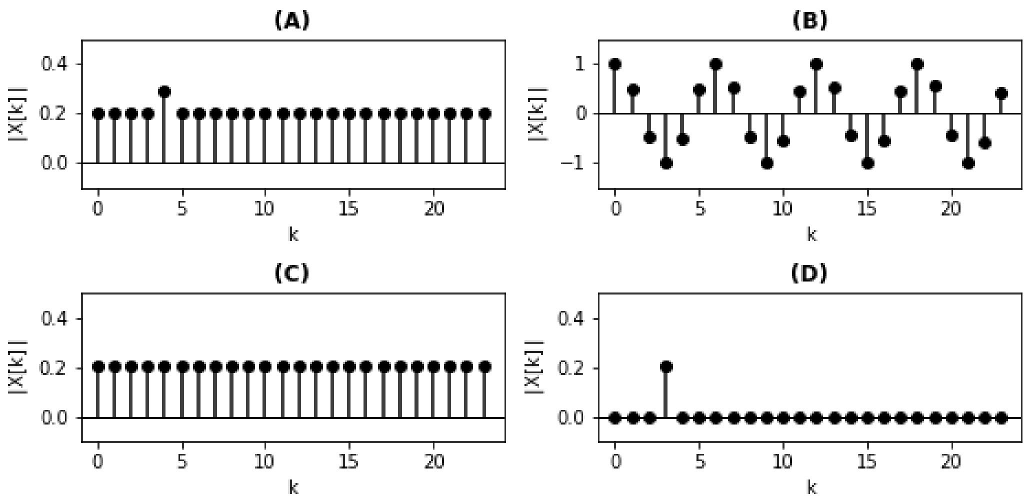


Answer

The correct DFT coefficients are shown in B. The new signal completes 3 full cycles during the discrete sequence. Its amplitude is the same as the first sequence.

b) Given the time domain signal below, which of the options below shows the correct DFT coefficient magnitudes?

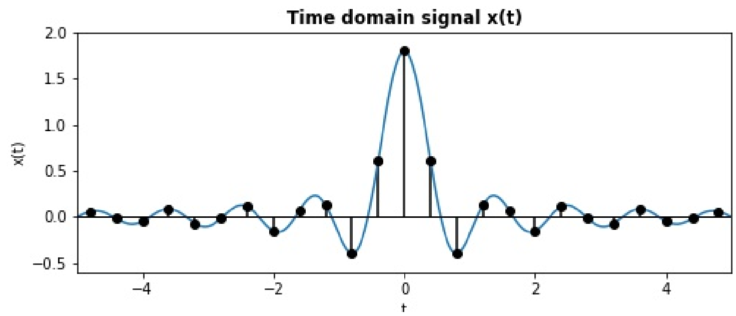


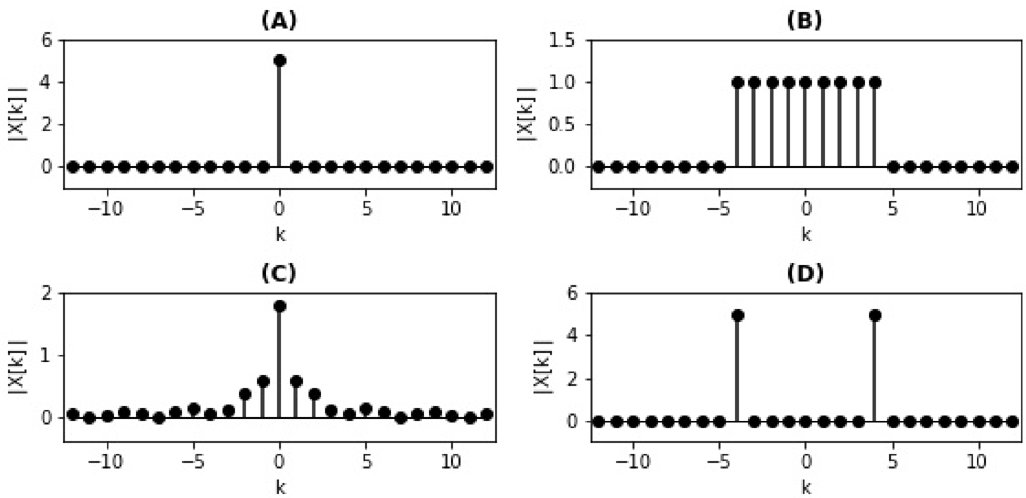


Answer

The correct DFT coefficients are shown in C. The DFT coefficients of a unit impulse are all $\frac{1}{\sqrt{N}}$. This impulse has been shifted, so the DFT coefficients have varying phase and are not purely real; however, their magnitudes are still uniformly $\frac{1}{\sqrt{N}}$.

c) Given the time domain signal below, which of the options below shows the correct DFT coefficient magnitudes?

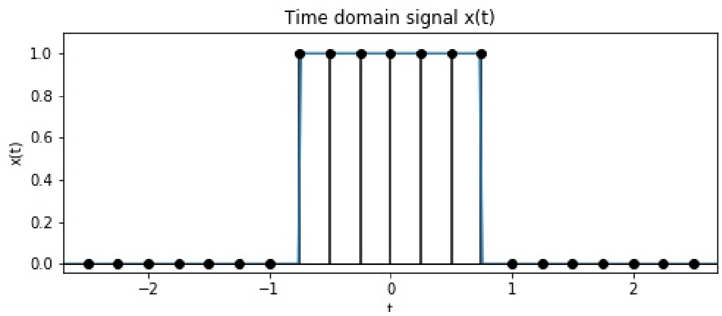


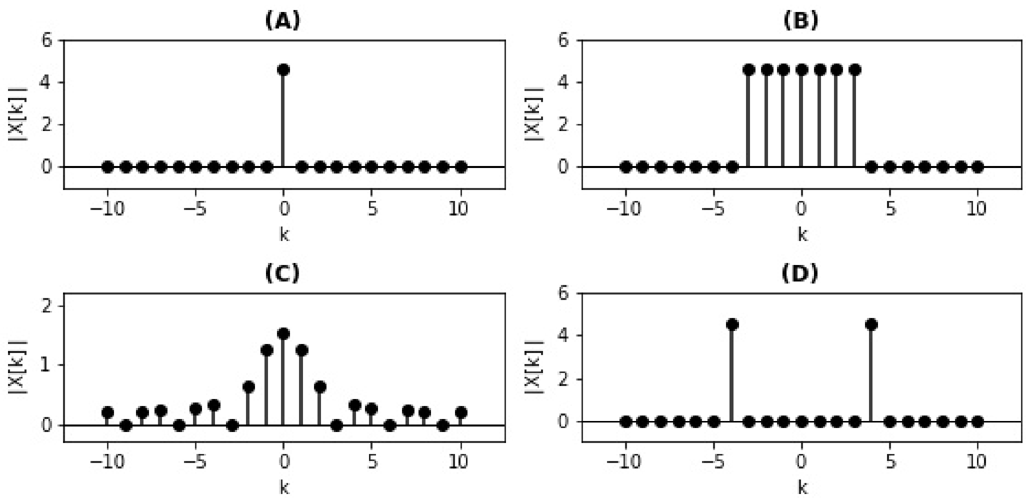


Answer

The correct DFT coefficients are shown in B. We saw in lecture that the DFT of a sinc is a boxcar and vice versa.

d) Given the time domain signal below, which of the options below shows the correct DFT coefficient magnitudes?





Answer

The correct DFT coefficients are shown in C. We saw in lecture that the DFT of a sinc is a boxcar and vice versa.