Discrete Fourier Transform

Assume we are working with an N length discrete signal and we would like to find its discrete frequencies. This is done through the Discrete Fourier Transform (DFT), which is simply a change of basis to the DFT basis.

First, let us vectorize our signal. If $x[n]$ is our input signal, we model it as a vector by letting the n^{th} coordinate be $x[n]$. In other words,

$$\vec{x} = \left[x[0], x[1], x[2], \ldots, x[N-1]\right]^T$$

In order to decompose $\vec{x}$ into its constituent frequencies, we must find the vector representation of these frequencies. A length N signal will have N different discrete frequencies of the following form. The fundamental frequency of the signal is called ω_N and its value is

$$\omega_N = e^{j\frac{2\pi}{N}} \tag{1}$$

The DFT Basis is built by taking powers of this fundamental frequency. We define the k^{th} basis vector $\vec{u}_k[n]$ as

$$\vec{u}_k[n] = \frac{1}{\sqrt{N}}\omega_N^{kn} \text{ for } k = 0, 1, \ldots N-1 \tag{2}$$

The matrix U has columns which consist of the N DFT basis vectors

$$U = \begin{bmatrix}\vec{u}_0 & \vec{u}_1 & \cdots & \vec{u}_{N-1}\end{bmatrix} \tag{3}$$

We choose to normalize all of these vectors by a factor of $\frac{1}{\sqrt{N}}$ so that the DFT basis vectors are orthonormal. Try to verify on your own that

$$\langle \vec{u}_p, \vec{u}_q \rangle = \sum_{n=0}^{N-1} \overline{u_q}[n] u_p[n] = \begin{cases} 0, & p \neq q \\ 1, & p = q \end{cases}$$

To represent a signal $x[n]$ in the frequency domain, we can change coordinates to the U basis. We define the matrix F as the matrix that takes our time-domain signal and transforms it into the frequency domain.

$$F = U^{-1} = U^* = \frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^{-1\cdot 1} & \omega_N^{-1\cdot 2} & \cdots & \omega_N^{-1\cdot(N-1)} \\ 1 & \omega_N^{-2\cdot 1} & \omega_N^{-2\cdot 2} & \cdots & \omega_N^{-2\cdot(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{-(N-1)\cdot 1} & \omega_N^{-(N-1)\cdot 2} & \cdots & \omega_N^{-(N-1)\cdot(N-1)} \end{bmatrix} \tag{4}$$

Similarly, the matrix U takes a signal $X[k]$ in the frequency domain and converts it back to the time-domain.

$$U = F^{-1} = F^* = \frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N^{1\cdot 1} & \omega_N^{1\cdot 2} & \cdots & \omega_N^{1\cdot(N-1)} \\ 1 & \omega_N^{2\cdot 1} & \omega_N^{2\cdot 2} & \cdots & \omega_N^{2\cdot(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{(N-1)\cdot 1} & \omega_N^{(N-1)\cdot 2} & \cdots & \omega_N^{(N-1)\cdot(N-1)} \end{bmatrix} \tag{5}$$

The relationship between a time-domain signal $x[n]$ and its frequency components $X[k]$ can be written as

$$x[n] = X[0]\vec{u}_0 + \ldots + X[N-1]\vec{u}_{N-1} = UX[k] \tag{6}$$

1 Roots of Unity

The DFT is a coordinate transformation to a basis made up of roots of unity. In this problem we explore some properties of the roots of unity. An Nth root of unity is a complex number z satisfying the equation $z^N = 1$ (or equivalently $z^N - 1 = 0$).

a) Show that $z^N - 1$ factors as

$$z^N - 1 = (z-1)(\sum_{k=0}^{N-1} z^k).$$

Answer

$$(z-1)(\sum_{k=0}^{N-1} z^k) = \sum_{k=1}^{N} z^k - \sum_{k=0}^{N-1} z^k = z^N - z^0 = z^N - 1$$

b) Show that any complex number of the form $\omega_k = e^{j\frac{2\pi}{N}k}$ for $k \in \mathbb{Z}$ is an N-th root of unity.

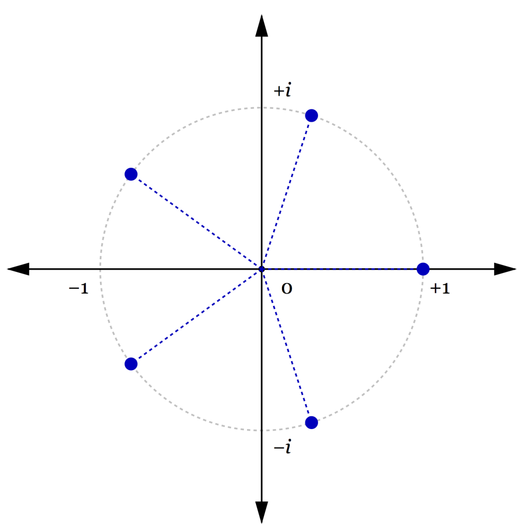
Answer

$$\omega_k^N = \left(e^{j\frac{2\pi}{N}k}\right)^N = e^{j2\pi k} = e^0 = 1$$

c) Draw the fifth roots of unity in the complex plane. How many of them are there?

Answer

There are 5 fifth roots of unity (and in general there are N Nth roots of unity).



d) Let $\omega_1 = e^{j\frac{2\pi}{5}}$. What is ω_1^2? What is ω_1^3? What is ω_1^{42}?

Answer

$$\omega_1^2 = \omega_2$$
$$\omega_1^3 = \omega_3$$
$$\omega_1^{42} = \omega_1^2 = \omega_2$$

e) What is the complex conjugate of ω_1? What is the complex conjugate of ω_{42}?

Answer

$$\bar{\omega}_1 = \omega_{-1} = \omega_4$$
$$\bar{\omega}_{42} = \omega_{-42} = \omega_3$$

f) Compute $\sum_{k=0}^{N-1} \omega^k$ where ω is some root of unity. Does the answer make sense in terms of the plot you drew?

Answer

If $\omega = 1$ then this is easy: we have

$$\sum_{k=0}^{N-1} \omega^k = \sum_{k=0}^{N-1} 1 = N.$$

If $\omega \neq 1$ then we can use the formula we found in part (a) to write

$$\sum_{k=0}^{N-1} \omega^k = \frac{\omega^N - 1}{\omega - 1} = 0$$

since ω is a root of unity. This makes sense because all the roots of unity are spaced evenly around the circle. Therefore summing them up, we get zero.

2 DFT of pure sinusoids

a) Consider the continuous-time signal $x(t) = \cos\left(\frac{2\pi}{3}t\right)$. Suppose that we sampled it every 1 second to get (for $n = 3$ time steps):

$$x[n] = \begin{bmatrix}\cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right)\end{bmatrix}^T.$$

Compute $\vec{X}[k]$ and the basis vectors $\vec{u}_k$ for this signal.

Answer

The basis vectors are

$$\vec{u}_0 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad \vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\e^{j\frac{2\pi}{3}}\\e^{j\frac{4\pi}{3}}\end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\e^{j\frac{4\pi}{3}}\\e^{j\frac{8\pi}{3}}\end{bmatrix}$$

The frequency components $\vec{X}[k]$ can be computed by multiplying by $F = U^*$.

$$\vec{X} = F\vec{x} = \frac{\sqrt{3}}{2}\begin{bmatrix}0 & 1 & 1\end{bmatrix}^T$$

b) Now for the same signal as before, suppose that we took $n = 6$ samples. In this case we would have:

$$x[n] = \begin{bmatrix}\cos\left(\frac{2\pi}{3}(0)\right) & \cos\left(\frac{2\pi}{3}(1)\right) & \cos\left(\frac{2\pi}{3}(2)\right) & \cos\left(\frac{2\pi}{3}(3)\right) & \cos\left(\frac{2\pi}{3}(4)\right) & \cos\left(\frac{2\pi}{3}(5)\right)\end{bmatrix}^T.$$

Repeat what you did above. What are $X[k]$ and the basis vectors $\vec{u}_k$ for this signal.

Answer

The basis vectors are

$$\vec{u}_k[n] = \frac{1}{\sqrt{6}}e^{j\frac{2\pi}{6}kn}$$

The frequency components $\vec{X}[k]$ can be computed by multiplying by $F = U^*$.

$$\vec{X} = F\vec{x} = \frac{\sqrt{6}}{2}\begin{bmatrix}0 & 0 & 1 & 0 & 1 & 0\end{bmatrix}^T$$

Alternatively we writing $x[n]$ as a linear combination of exponentials

$$x[n] = \frac{1}{2}e^{j\frac{2\pi}{3}n} + \frac{1}{2}e^{-j\frac{2\pi}{3}n} = \frac{\sqrt{6}}{2}\vec{u}_2 + \frac{\sqrt{6}}{2}\vec{u}_4$$

This tells us that $X[0] = X[1] = X[3] = X[5] = 0$ and $X[2] = X[4] = \frac{\sqrt{6}}{2}$.

c) Let's do this more generally. For the signal $x(t) = \cos\left(\frac{2\pi m}{N}t\right)$, where m is an integer between 0 and $N-1$, compute the frequency components $X[k]$ where $x[n]$ is a time-domain signal of length N.

$$x[n] = \begin{bmatrix}\cos\left(\frac{2\pi m}{N}(0)\right) & \cos\left(\frac{2\pi m}{N}(1)\right) & \cdots & \cos\left(\frac{2\pi m}{N}(N-1)\right)\end{bmatrix}^T.$$

Answer

The basis vectors are

$$\vec{u}_k[n] = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}kn}$$

Writing out the $x[n]$ as a linear combination of exponentials

$$x[n] = \frac{1}{2}e^{j\frac{2\pi m}{N}n} + \frac{1}{2}e^{-j\frac{2\pi m}{N}n}$$

This tells us that

$$X[m] = X[N-m] = \frac{\sqrt{N}}{2}$$
$$X[k] = 0 \text{ for } k \neq m, N-m.$$

If $m = 0$, then $X[0] = \sqrt{N}$ and all other entries are 0.