1 Controller Canonical Form

When working with systems in state-space, you may have noticed that a single system can be represented in many different forms, depending on factors, such as how you ordered your state vector. Writing out systems in certain **canonical forms** often allows engineers to quickly determine system behavior.

The **controller canonical form**, which guarantees controllability and simplifies eigenvalue placement, takes on the following form:

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{bmatrix} \tag{1}$$

Change of Basis to Controller Canonical Form

Given a **controllable** system of the form $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t)$, we can transform it into controller canonical form by choosing some T, such that:

$$\vec{z} = T\vec{x} \qquad \tilde{A} = TAT^{-1} \qquad \tilde{B} = TB$$

for matrices $\tilde{A}$ and $\tilde{B}$ of the form shown above.

We can calculate this T using $C = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$, the controllability matrix of the original form using A and B. Note that C is full rank, and therefore invertible, because the original system is controllable. We saw in lecture how to construct this matrix T by taking the last row of C^{-1} as $\vec{q}^T$.

$$T = \begin{bmatrix} - & \vec{q}^T & - \\ - & \vec{q}^T A & - \\ & \vdots & \\ - & \vec{q}^T A^{n-1} & - \end{bmatrix}$$

However, let us show a more concrete formula for this transformation T by computing the controllability matrix in the controller basis $\vec{z}$.

$$\begin{aligned} \tilde{C} &= \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix} = \begin{bmatrix} TB & TAT^{-1}TB & \cdots & TA^{n-1}T^{-1}TB \end{bmatrix} \\ &= T\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = TC \implies T = \tilde{C}C^{-1} \end{aligned}$$

Also notice that when we place our system in feedback using $u(t) = -\tilde{K}\vec{z} = -K\vec{x}$ with $\tilde{K} = KT^{-1}$, we get the closed loop matrix

$$T(A - BK)T^{-1} = \tilde{A} - \tilde{B}\tilde{K}$$

The eigenvalues of both systems are the same, and we can arbitrarily assign the eigenvalues of $\tilde{A} - \tilde{B}\tilde{K}$ with the choice of $\tilde{K}$. Therefore, we just proved that *controllability enables arbitrary eigenvalue assignment* in any state space system. Note that it is *not* necessary to bring the system to the controller canonical form to assign its eigenvalues. You can still use what we did in the last section to choose K in order to obtain desirable eigenvalues.

2 Eiegenvalue Placement in CCF

Consider the following continuous-time system

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

a) Is this system controllable?

Answer

The system is in Controllable Canonical Form meaning it must be controllable. Alternatively we can compute the controllability matrix

$$C = [B, AB, A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix}$$

Observe that C matrix is full rank and hence our system is controllable.

b) Is the linear continuous time system stable?

Answer

Because the matrix A is in controllable canonical form, we can find the characteristic polynomial to be $\lambda^3 + 4\lambda^2 + 3\lambda$. From that polynomial we calculate the eigenvalues of matrix A:

$$0 = \lambda^3 + 4\lambda^2 + 3\lambda = \lambda(\lambda + 3)(\lambda + 1)$$

The eigenvalues are then $0, -3, -1$. Since one of the eigenvalues has zero real part, this system is unstable.

c) Using state feedback $u(t) = -K\vec{x}(t) = \begin{bmatrix} -k_0 & -k_1 & -k_2 \end{bmatrix} \vec{x}(t)$ place the eigenvalues at $-1, -1, -2$.

Answer

The closed loop system is given by

$$A + BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_0 & -k_1 - 3 & -k_2 - 4 \end{bmatrix}$$

which has characteristic polynomial $\lambda^3 + (4 + k_2)\lambda^2 + (3 + k_1)\lambda + k_0$. To place the eigenvalues at $-1, -1, -2$, the desired characteristic polynomial is $(\lambda+1)(\lambda+1)(\lambda+2) = \lambda^3 + 4\lambda^2 + 5\lambda + 2$. So we should choose $k_0 = 2, k_1 = 2, k_2 = 0$.

3 Controllable Canonical Form - Eigenvalues Placement

Consider the following linear discrete time system

$$\vec{x}[t+1] = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u[t]$$

a) Is this system controllable?

Answer

We calculate

$$C = [B, AB, A^2B] = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Observe that C matrix is full rank and hence our system is controllable.

b) Is the linear discrete time system stable?

Answer

We have to calculate the eigenvalues of matrix A. Thus,

$$0 = \det(A - \lambda I) = \lambda^3 + 4\lambda^2 + 3\lambda = \lambda(\lambda + 3)(\lambda + 1)$$

Since the eigenvalue at -3 is outside the unit circle, this system is unstable.

c) Bring the system to the controllable canonical form

$$\vec{z}[t+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_0 & a_1 & a_2 \end{bmatrix} \vec{z}[t] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[t]$$

using transformation $\vec{z}[t] = T\vec{x}[t]$

Answer

The characteristic polynomial for this system is $\lambda^3 + 4\lambda^2 + 3\lambda$. So putting this system in control canonical form gives

$$\vec{z}[t+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix} \vec{z}[t] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[t]$$

To compute the transformation T, we need the matrices

$$C = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \ldots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix}$$

Then T is given by

$$T = \tilde{C}C^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -4 \\ 1 & -4 & 13 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix}$$

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d) Using state feedback $u[t] = \tilde{K}\vec{z}[t] = \begin{bmatrix} \tilde{k}_0 & \tilde{k}_1 & \tilde{k}_2 \end{bmatrix} \vec{z}[t]$ place the eigenvalues at $0, 1/2, -1/2$.

Answer

The closed loop system in z coordinates is given by

$$\tilde{A} + \tilde{B}\tilde{K} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_1 & k_2 - 3 & k_3 - 4 \end{bmatrix}$$

which has characteristic polynomial $\lambda^3 + (4 - k_3)\lambda^2 + (3 - k_2)\lambda - k_1$. To place the eigenvalues at 0, 1/2, -1/2, the desired characteristic polynomial is $\lambda(\lambda - 1/2)(\lambda + 1/2) = \lambda^3 - 1/4\ \lambda$. So we should choose $\tilde{k}_0 = 0, \tilde{k}_1 = 13/4, \tilde{k}_2 = 4$.

e) Convert your controller back into the standard basis so that $u[t] = K\vec{x}[t]$.

Answer

If we want the feedback controller in terms of $x[t]$, we write $u[t] = \tilde{K}\vec{z}[t] = \tilde{K}T\vec{x}[t]$. This tells us that

$$K = \tilde{K}T = \begin{bmatrix} 0 & 13/4 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -35/4 & 19/4 \end{bmatrix}.$$