1 Airplane Discretization

In this question we will explore briefly a simplified linear model of an airplane control model. First let's define the variables we will be working with, in reference with figure 1:

(i) α = Angle of Attack (angle the plane makes with the direction of wind)

(ii) θ = Pitch Angle (angle of the plane with respect to the horizontal)

(iii) u = Elevator angle (used to control the aircraft's pitch)

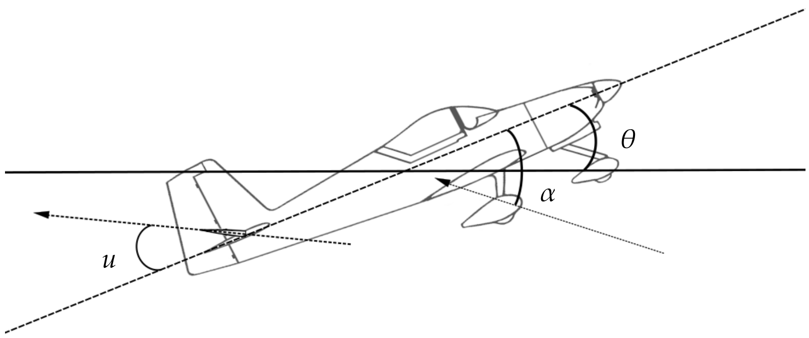


Figure 1: Simplified Airplane model

Now, consider the following (simplified) continuous time system:

$$\frac{d}{dt}\alpha = 5\alpha - \frac{d}{dt}\theta + c_1\delta \tag{1}$$

$$\frac{d^2}{dt^2}\theta = -\alpha + \frac{d}{dt}\theta + c_2\delta \tag{2}$$

a) For the system of differential equations given, **write the matrix differential equation as**

$$\frac{d}{dt}\vec{x} = A\vec{x} + B\delta$$

Answer

We can define our state vector as $\vec{x} = \begin{bmatrix} \alpha \\ \theta \\ \frac{d\theta}{dt} \end{bmatrix}$ to get the equations

$$\frac{d}{dt}x_1 = 5x_1 - x_3 + c_1\delta$$

$$\frac{d}{dt}x_2 = x_3$$

$$\frac{d}{dt}x_3 = -x_1 + x_3 + c_2\delta$$

This gives us the following state-matrices

$$A = \begin{bmatrix} 5 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} c_1 \\ 0 \\ c_2 \end{bmatrix} \tag{3}$$

Alternatively since the state θ is never used, we can pick $\vec{x} = \begin{bmatrix} \alpha \\ \frac{d\theta}{dt} \end{bmatrix}$ with the following state-matrices.

$$A = \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{4}$$

b) Now, assume for some specific component values we get the following differential equation:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t). \tag{5}$$

Unfortunately, we are unable to measure our state vector continuously. Suppose that we sample the system with some sampling interval T. Let us discretize the above system. Assume that we use piecewise constant voltage inputs $u(t) = u[k]$ for $t \in [kT, (k+1)T)$.

Calculate a discrete-time system for Equation (5)'s continuous-time vector system in the form:

$$\vec{x}[k+1] = A_d\vec{x}[k] + \vec{b}_d[k].$$

Answer

One way to discretize this system is to change coordinates to the eigenbasis, and discretize the individual scalar equations. Having done so, we can change coordinates back to the standard basis.

Since A is diagonalizable, we can write $A = V\Lambda V^{-1}$, substitute into our differential equation, to get:

$$\begin{aligned} \frac{d}{dt}\vec{x} &= V\Lambda V^{-1}\vec{x} + \vec{b}u(t) \\ \frac{d}{dt}V^{-1}\vec{x} &= \Lambda V^{-1}\vec{x} + V^{-1}\vec{b}u(t) \\ \frac{d}{dt}\vec{z} &= \Lambda\vec{z} + V^{-1}\vec{b}u(t) \end{aligned}$$

Writing, $\vec{z} = V^{-1}\vec{x}$, we can diagonlize the system. We can also compute the following matrices

$$V = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \qquad V^{-1}\vec{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

This gives us two differential equations

$$\begin{aligned} \frac{d}{dt}z_1(t) &= -2z_1(t) - 2u(t) \\ \frac{d}{dt}z_2(t) &= -z_2(t) - 2u(t) \end{aligned}$$

Recall that the following scalar differential equation can be discretized as the following

$$\frac{d}{dt}x(t) = \lambda x(t) + bu(t) \implies x[k+1] = e^{\lambda T}x[k] + b\frac{e^{\lambda T}-1}{\lambda}u[k]$$

Hence, we can discretize the system in this diagonal space, giving us

$$\begin{aligned} z_1[k+1] &= e^{-2T}z_1[k] + (e^{-2T}-1)u[k] \\ z_2[k+1] &= e^{-T}z_2[k] + 2(e^{-T}-1)u[k] \end{aligned}$$

This gives us the equation

$$\vec{z}[k+1] = \begin{bmatrix} e^{-2T} & 0 \\ 0 & e^{-T} \end{bmatrix}\vec{z}[k] + \begin{bmatrix} e^{-2T}-1 \\ 2e^{-T}-2 \end{bmatrix}u[k].$$

Changing coordinates back to $\vec{x}$ we see that

$$\begin{aligned} A_d &= \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} e^{-2T} & 0 \\ 0 & e^{-T} \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \\ \vec{b}_d &= \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} e^{-2T}-1 \\ 2e^{-T}-2 \end{bmatrix} \end{aligned}$$

Multiplying out the matrices, we get:

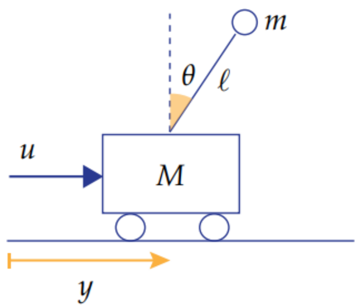
$$\begin{aligned} A_d &= \begin{bmatrix} 2e^{-T}-e^{-2T} & e^{-T}-e^{-2T} \\ 2e^{-2T}-2e^{-T} & 2e^{-2T}-e^{-T} \end{bmatrix} \\ \vec{b}_d &= \begin{bmatrix} e^{-2T}-2e^{-T}+1 \\ 2e^{-T}-2e^{-2T} \end{bmatrix} \end{aligned}$$

2 Inverted Pendulum on a Rolling Cart

Recall the inverted pendulum depicted below from Homework 6 problem 1, which is placed on a rolling cart and whose equations of motion are given by:

$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{u}{m} + \dot{\theta}^2 \ell \sin\theta - g\sin\theta\cos\theta\right)$$
$$\ddot{\theta} = \frac{1}{\ell(\frac{M}{m} + \sin^2\theta)}\left(-\frac{u}{m}\cos\theta - \dot{\theta}^2 \ell \cos\theta\sin\theta + \frac{M+m}{m} g\sin\theta\right).$$

where we use $\dot{x}$ to denote the time derivative of x; that is, $\dot{y} = \frac{dy}{dt}$, $\dot{\theta} = \frac{d\theta}{dt}$, $\ddot{y} = \frac{d^2y}{dt^2}$ and $\ddot{\theta} = \frac{d^2\theta}{dt^2}$.



a) Recall the result of our linearized model from Homework 6, problem 1c:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \approx \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ \frac{M+m}{lM} g & 0 & 0 \\ -\frac{m}{M} g & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -\frac{1}{lM} \\ \frac{1}{M} \end{bmatrix}}_{B} u.$$

Show that the linearized model is controllable.

Answer

Observe that

$$AB = \begin{bmatrix} -\frac{1}{lM} \\ 0 \\ 0 \end{bmatrix}$$

and

$$A^2B = \begin{bmatrix} 0 \\ -\frac{M+m}{(lM)^2} g \\ \frac{m}{lM^2} g \end{bmatrix}$$

Then,

$$C = \begin{bmatrix} 0 & -\frac{1}{lM} & 0 \\ -\frac{1}{lM} & 0 & -\frac{M+m}{(Ml)^2} g \\ \frac{1}{M} & 0 & \frac{m}{lM^2} g \end{bmatrix}$$

Since we are trying to test rank, we can remove scalar terms from the vectors. We then get,

$$\text{rank } C = \text{rank} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ l & 0 & \left(\frac{m}{M+m}\right) l \end{bmatrix}$$

We want to show that,

$$\frac{m}{M+m} \neq 1$$

This must be the case because this is only true when $M = 0$, which is not possible since an object at this macro scale must have mass.

b) Suppose $M = 1$, $m = 0.1$, $l = 1$, and $g = 10$, and design a state feedback controller,

$$u(t) = -k_1\theta(t) - k_2\dot{\theta}(t) - k_3\dot{y}(t),$$

such that the eigenvalues of $A - BK$ (the "closed-loop eigenvalues") are $\lambda_1 = \lambda_2 = \lambda_3 = -1$.

Answer

Plugging in values, the system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 11 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} u$$

Setting $u = -K\vec{x}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 \\ 11 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 11 + k_1 & k_2 & k_3 \\ -k_1 - 1 & -k_2 & -k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The characteristic polynomial is

$$p_{(A-BK)}(\lambda) = \lambda^3 + \lambda^2(k_3 - k_2) + \lambda(-k_1 - 11) - 10k_3 = 0$$

Our target polynomial is

$$p_{(A-BK)}(\lambda) = \lambda^3 + 3\lambda^2 + 3\lambda + 1$$

Comparing coefficients, we get

$$k_1 = -14, k_2 = -3.1, k_3 = -0.1$$

c) Suppose we set $k_2 = k_3 = 0$ and vary only k_1; that is, the controller uses only $\theta(t)$ for feedback. Does there exist a k_1 value such that all closed-loop eigenvalues have negative real parts?

Answer

The characteristic polynomial is

$$p_{(A-BK)}(\lambda) = \lambda^3 + \lambda(-k_1 - 11) = 0$$

No matter what k_1 is, there will always be an eigenvalue at 0.

3 Minimum Norm Control

Suppose we had a linear discrete-time system with the following dynamics

$$\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{6}$$

Given the initial state $\vec{x}[0] = \vec{0}$, we would like to reach a target state $\vec{x}_t = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

a) Show that we can reach the state $\vec{x}_t$ in a finite number of time-steps

Answer

If the system is controllable, then we can reach $\vec{x}_t$ in a finite number of time-steps. The controllability matrix can be computed accordingly.

$$C = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

Since the controllability matrix is of full-rank, we can reach any state $\vec{x} \in \mathbb{R}^2$ in at most two time-steps.

b) What sequence of control inputs can we give to reach the state $\vec{x}_t$ in two time-steps?

Answer

We can write out $\vec{x}[2]$ in terms of the previous states to say

$$\begin{aligned} \vec{x}[2] &= A\vec{x}[1] + \vec{b}u[1] = A^2\vec{x}[0] + A\vec{b}u[0] + \vec{b}u[1] \\ &= \begin{bmatrix} \vec{b} & A\vec{b} \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix} = C \begin{bmatrix} u[1] \\ u[0] \end{bmatrix} \end{aligned}$$

Therefore we can solve for $u[0]$ and $u[1]$ by taking the inverse of C

$$\begin{bmatrix} u[1] \\ u[0] \end{bmatrix} = C^{-1}\vec{x}[2] = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$$

c) While we can theoretically reach $\vec{x}_t$ in two time-steps, we notice that it is too expensive to move our system this quickly. Set up an optimization problem of the form below to reach $\vec{x}_t$ in five time-steps with minimum energy.

$$\min_{\vec{w} \in \mathbb{R}^5} \left\| \vec{w} \right\|^2 \text{ subject to } H\vec{w} = \vec{y} \tag{7}$$

Answer

Let's start by writing out $\vec{x}[5]$ in terms of the previous states

$$\begin{aligned} \vec{x}[5] &= A^5\vec{x}[0] + A^4\vec{b}u[0] + A^3\vec{b}u[1] + A^2\vec{b}u[2] + A\vec{b}u[3] + \vec{b}u[4] \\ &= \begin{bmatrix} A^4\vec{b} & A^3\vec{b} & A^2\vec{b} & A\vec{b} & \vec{b} \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ u[3] \\ u[4] \end{bmatrix} \\ \vec{y} &= H\vec{w} \end{aligned}$$

Since $H = \begin{bmatrix} A^4\vec{b} & A^3\vec{b} & A^2\vec{b} & A\vec{b} & \vec{b} \end{bmatrix}$ is a wide matrix of full-rank, we see that the system of equations $H\vec{w} = \vec{y}$ has infinite solutions. Our goal is to reach $\vec{x}_t$ with minimum energy or in otherwords, we would like to minimize $\|\vec{w}\|^2 = w_1^2 + \ldots + w_5^2$ over all solutions of $H\vec{w} = \vec{y}$.

This is equivalent to an optimization problem of the form

$$\min_{\vec{w} \in \mathbb{R}^5} \|\vec{w}\|^2 \text{ subject to } H\vec{w} = \vec{y}$$

d) What is the solution to the optimization problem from the previous part? Give the optimal solution $\vec{w}^*$.

Answer

We saw from HW and lecture that the optimal solution to the problem can be computed through the pseudo-inverse

$$\vec{w}^* = H^\dagger \vec{y} = H^T (HH^T)^{-1} \vec{y}$$

We can plug this into numpy to get the following optimal solution

$$\vec{w}^* = \begin{bmatrix} 4 & -3.6 & 2.8 & -1.3 & -1.75 \end{bmatrix}^T$$

e) Try solving the minimum norm problem for 2 steps all the way up to 10 steps and compare the norms of each solution.

Answer

Notice how the norm drops as we increase the number of time-steps. This is because we are relaxing the constraints of the problem and giving the system more freedom to reach our target in the allotted time.