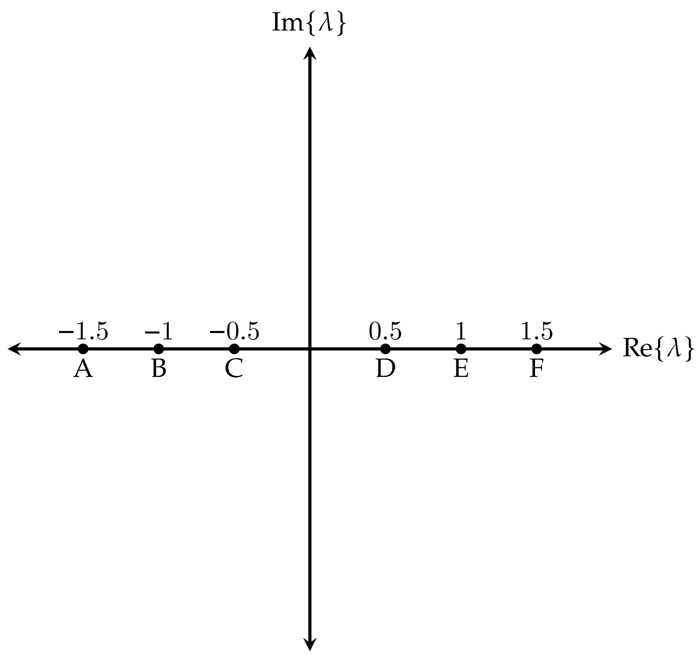
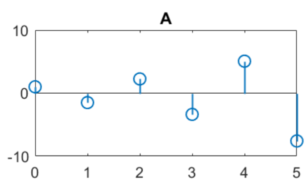
1 Discrete time system responses

We have a system $x[k+1] = \lambda x[k]$. For each λ value plotted on the real-imaginary axis, sketch $x[k]$ with an initial condition of $x[0] = 1$. Determine if each system is stable.



Answer



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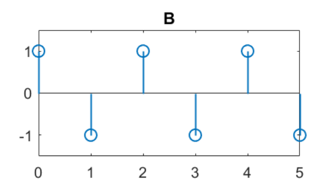

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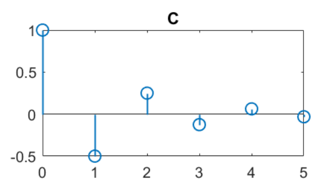

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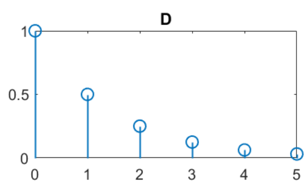

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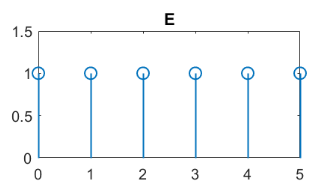

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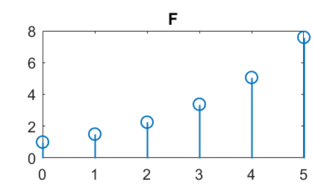

Unstable

2 Eigenvalues Placement in Discrete Time

Consider the following linear discrete time system

$$\vec{x}[t+1] = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t] + \vec{w}[t] \tag{1}$$

a) Is this system controllable?

Answer

We calculate

$$C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Observe that C matrix is full rank and hence our system is controllable.

b) Is the linear discrete time system stable?

Answer

We have to calculate the eigenvalues of matrix A. Thus,

$$\det(\lambda I - A) = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -2$$

Since the magnitudes of the eigenvalues λ_1 and λ_2 are greater than 1, the system is unstable.

c) Derive a state space representation of the resulting closed loop system using state feedback of the form $u[t] = \begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}[t]$

Answer

The closed loop system using state feedback has the form

$$\begin{aligned} \vec{x}[t+1] &= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t] \\ &= \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \left(\begin{bmatrix} k_1 & k_2 \end{bmatrix} \vec{x}[t] \right) \\ &= \left\{ \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right\} \vec{x}[t] \end{aligned}$$

Thus, the closed loop system has the form

$$\vec{x}[t+1] = \underbrace{\begin{bmatrix} k_1 & 1+k_2 \\ 2 & -1 \end{bmatrix}}_{A_{\text{cl}}} \vec{x}[t]$$

d) Find the appropriate state feedback constants, k_1, k_2 in order the state space representation of the resulting closed loop system to place the eigenvalues at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$

Answer

$k_1 = 1, k_2 = -\frac{11}{8}$

Answer

From the previous part we have computed the closed loop system as

$$\vec{x}[t+1] = \underbrace{\begin{bmatrix} k_1 & 1+k_2 \\ 2 & -1 \end{bmatrix}}_{A_{\text{cl}}} \vec{x}[t]$$

Thus, finding the eigenvalues of the above system we have

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} k_1 - \lambda & 1+k_2 \\ 2 & -1-\lambda \end{bmatrix} = \lambda^2 + (1-k_1)\lambda + (-k_1 - 2k_2 - 2)$$

We want to place the eigenvalue at $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{2}$. This means that we should choose the gains k_1 and k_2 so that the characteristic equation is

$$0 = (\lambda - \frac{1}{2})(\lambda + \frac{1}{2}) = \lambda^2 - \frac{1}{4}.$$

Thus we should choose k_1 and k_2 satisfying the system of equations

$$\begin{aligned} 0 &= 1 - k_1 \\ -\frac{1}{4} &= -k_1 - 2k_2 - 2 \end{aligned}.$$

This system has solution $k_1 = 1, k_2 = -\frac{11}{8}$.

e) Suppose that instead of $\begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t]$ in (1), we had $\begin{bmatrix} 1 \\ 1 \end{bmatrix} u[t]$ as the way that the discrete-time control acted on the system. Is this system controllable from $u[t]$?

Answer

We calculate

$$C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Observe that C matrix is not full rank and hence our system is not controllable.

f) For the part above, suppose we used $[k_1, k_2]$ to try and control the system. What would the eigenvalues be? Can you move all the eigenvalues to where you want? Give an intuitive explanation of what is going on.

Answer

$$\vec{x}[t+1] = \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) \vec{x}[t] \tag{2}$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ k_1 & k_2 \end{bmatrix} \right) \vec{x}[t] \tag{3}$$

$$\tag{4}$$

Finding the eigenvalues λ:

$$
\begin{aligned}
\det\left(\begin{bmatrix} k_1-\lambda & k_2+1 \\ k_1+2 & k_2-1-\lambda \end{bmatrix}\right) &= 0 && (5)\\
&= (k_1-\lambda)(k_2-1-\lambda)-(k_1+2)(k_2+1) && (6)\\
&= k_1(k_2-1)-k_1\lambda-\lambda(k_2-1)+\lambda^2-(k_1k_2+k_1+2k_2+2) && (7)\\
&= k_1k_2-k_1-k_1\lambda-\lambda k_2+\lambda+\lambda^2-k_1k_2-k_1-2k_2-2 && (8)\\
&= \lambda^2+(1-k_1-k_2)\lambda-2(1+k_1+k_2) && (9)\\
&= (\lambda+2)(\lambda-(1+k_1+k_2)) && (10)
\end{aligned}
$$

We can see that the eigenvalue at $\lambda=-2$ cannot be moved, so we cannot arbitrarily change our eigenvalues with this control input.

3 Eigenvalue Placement

Consider the following linear discrete time system

$$\vec{x}(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -9 & -6 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

a) **What are the eigenvalues of the A matrix? Is this system stable?**

Answer

We can compute the eigenvalues through the characteristic polyomial

$$\chi_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -9 & -6-\lambda \end{bmatrix} = -\lambda \begin{vmatrix} -\lambda & 1 \\ -9 & -6-\lambda \end{vmatrix} = -\lambda(\lambda^2 + 6\lambda + 9) = 0$$

The eigenvalues are at $\lambda = 0, -3$ meaning the system is unstable since -3 is outside of the unit circle.

b) **Using state feedback $u(t) = K\vec{x}(t) = \begin{bmatrix} k_0 & k_1 & k_2 \end{bmatrix} \vec{x}(t)$ place the eigenvalues at $0, 1/2, -1/2$.**

Answer

The characteristic polynomial for the feedback matrix $A + BK$ will be:

$$\lambda^3 - (-6 + k_2)\lambda^2 - (-9 + k_1)\lambda - k_0 = 0$$

In order to place the eigenvalues at $0, \frac{1}{2}, -\frac{1}{2}$ we desire the characteristic polynomial:

$$\lambda^3 - \frac{1}{4}\lambda = 0$$

Therefore by matching coefficients, we see that

$$6 - k_2 = 0$$
$$9 - k_1 = -\frac{1}{4}$$
$$k_0 = 0$$

So it follows that

$$k_0 = 0, k_1 = \frac{37}{4}, k_2 = 6$$

c) **Suppose we now have a limitation on how much our controller can amplify $\vec{x}$ and all of our k values must be in between -5 and 5. Is it possible to pick a set of eigenvalues that will make the system stable?**

Answer

If all of our eigenvalues were on the unit circle at $\lambda = -1$, then the characteristic polynomial would be

$$(\lambda + 1)^3 = \lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$$

If we picked eigenvalues inside the unit circle, the coefficients to the characteristic polynomial would be smaller.

This tells us that the coefficients of the characteristic polynomial must be less than 3. However, if we try to set the λ coefficient to be less than 3, that would require $k_1 > 6$ which is not possible with our controller. Therefore, we cannot stabilize this system.