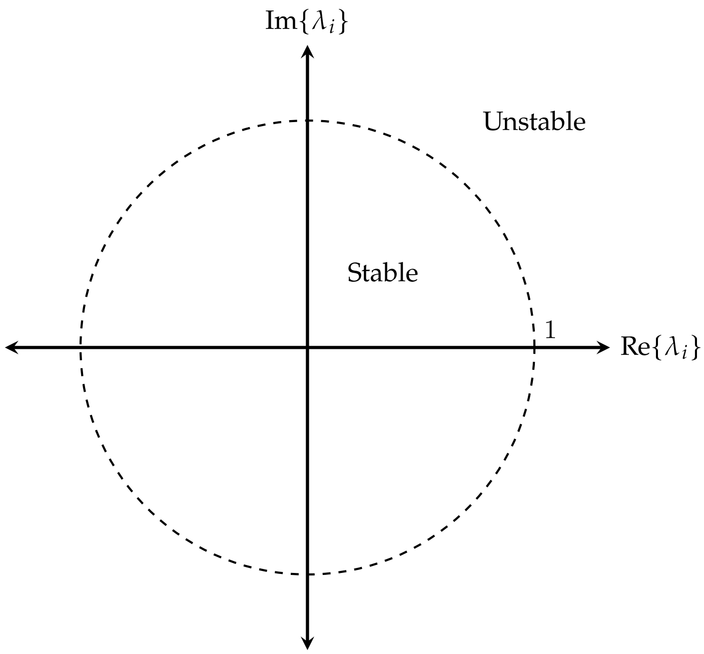
1 Stability

Discrete time systems

A discrete time system is of the form:

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$

This system is stable if $|\lambda_i| < 1$ for all λ_i, where λ_i's are the eigenvalues of A. If we plot all λ_i for A on the complex plane, if all λ_i lie within (not on) the unit circle, then the system is stable.



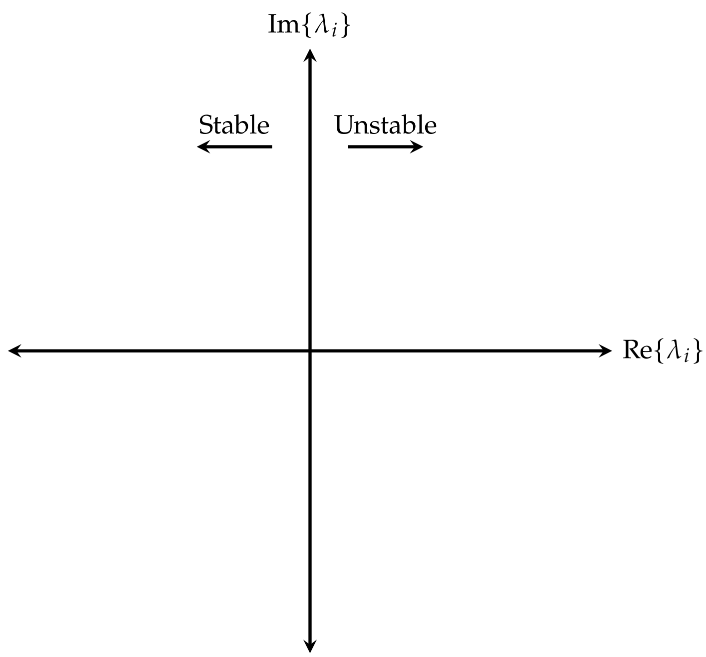
If $|\lambda| = 1$, we say the system is marginally stable and with respect to bounded-input bounded-output stability, this system would be unstable.

Continuous time systems

A continuous time system is of the form:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}(t) = A\vec{x}(t) + B\vec{u}(t)$$

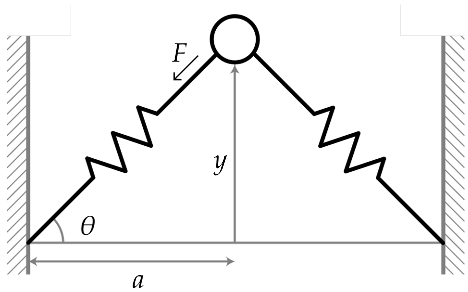
This system is stable if $\mathrm{Re}\{\lambda_i\} < 0$ for all λ_i , where λ_i's are the eigenvalues of A. If we plot all λ_i for A on the complex plane, if all λ_i lie to the left of $\mathrm{Re}\{\lambda_i\} = 0$, then the system is stable.



If $\mathrm{Re}\{\lambda_i\} = 0$, the system is marginally stable and it is again unstable with respect to bounded-input bounded-output stability.

2 Stability in continuous time system

Remember the spring-mass system introduced in Discussion 8A:



We assumed that each spring is linear with spring constant k and resting length X_0. The differential equation modeling this system was $\frac{d^2y}{dt^2} = -\frac{2k}{m}(y - X_0\frac{y}{\sqrt{y^2+a^2}})$. We built a state space model that describes how the displacement y of the mass from the spring base evolves. The state variables were $x_1 = y$ and $x_2 = \dot{y}$. Then we linearized the model around the equilibrium point $(x_1, x_2) = (0, 0)$, assuming $X_0 < a$. The linearized model is presented below.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{2k}{m}\left(1 - \frac{X_0}{a}\right) & 0 \end{bmatrix} x.$$

Compute the eigenvalues of your linearized model. Is this equilibrium stable?

Answer

To compute the eigenvalues, we solve

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -\frac{2k}{m}\left(1 - \frac{X_0}{a}\right) & -\lambda \end{bmatrix}\right) = \lambda^2 + \frac{2k}{m}\left(1 - \frac{X_0}{a}\right).$$

Since $X_0 < a$, this means that $\left(1 - \frac{X_0}{a}\right) > 0$. So we have a pair of imaginary eigenvalues

$$\lambda = \pm\sqrt{\frac{2k}{m}\left(1 - \frac{X_0}{a}\right)}j.$$

Since the linearized system has purely imaginary eigenvalues that are not repeated, their real parts are zero. Therefore the equilibrium is unstable.

3 Stability in discrete time system

Determine which values of α and β will make the following discrete-time state space models stable. Assume, α and β are real numbers and $b \neq 0$.

a)
$$x[t+1] = \alpha x[t] + bu(t)$$

Answer

$$|\alpha| < 1$$

b)
$$\vec{x}[t+1] = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \vec{x}[t] + b\vec{u}(t)$$

Answer

The eigenvalues of this system are:

$$\lambda = \alpha \pm j\beta$$

$$|\lambda| = \sqrt{\alpha^2 + \beta^2}$$

For this system to be stable, $|\lambda| < 1$, so

$$\alpha^2 + \beta^2 < 1$$

c)
$$\vec{x}[t+1] = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \vec{x}[t] + b\vec{u}(t)$$

Answer

The eigenvalues of this system are

$$\lambda = 1, 1$$

This means that regardless of α, this system is always unstable since $|\lambda| \geq 1$.