1 Compact SVD

Recall that an $m \times n$ matrix A will always have a singular value decomposition.

$$A = U\Sigma V^T \tag{1}$$

If A is of rank k, we can write out the SVD in **compact form** as $A = U_c \Sigma_c V_c^T$

$$A = U_c \Sigma_c V_c^T = \sum_{i=1}^{k} \sigma_i \vec{u}_i \vec{v}_i^T \tag{2}$$

where the subscript c is referring to the first k columns of the matrix. U_c is an $m \times k$ matrix, Σ_c is a $k \times k$ matrix and V_c^T is a $k \times n$ matrix.

If $m > n$ and A is a tall matrix of rank n, we can expand the matrices in block form to get the compact SVD where $V = V_c$

$$A = U\Sigma V^T = \begin{bmatrix} U_c & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_c \\ 0 \end{bmatrix} V^T = U_c \Sigma_c V_c^T$$

While if $m < n$, and A is a wide matrix of rank m, then $U_c = U$ and its compact SVD is

$$A = U\Sigma V^T = U \begin{bmatrix} \Sigma_c & 0 \end{bmatrix} \begin{bmatrix} V_c^T \\ V_2^T \end{bmatrix} = U_c \Sigma_c V_c^T$$

Recall that the columns of U_c span Col (A) while the columns of V_2 span Nul (A). Note that the matrix Σ_c is a diagonal matrix with the singular values on its diagonal.

2 Minimum Energy Warmup

Consider an undetermined system of equations given by

$$\vec{y} = A\vec{x}$$

where $A \in \mathbb{R}^{m \times n}$ is a wide matrix, that is $m < n$. We assume that A has linearly independent rows.

a) What is the rank of A?

Answer

The rows of A form a basis for Col(A^T) as they are assumed to be linearly independent. Thus, rank$(A) = m$ since rank$(A) = $ rank$(A^T) = $ dim Col(A^T).

b) In lecture, we saw that the minimum norm solution for $\vec{x}$ is given by

$$\vec{x} = V_c \Sigma_c^{-1} U_c^T \vec{y} = V_c \Sigma_c^{-1} U^T \vec{y}$$

where the definition of V_c and Σ_c come from the block matrix form of the SVD. That is,

$$A = U\Sigma V^T = U \begin{bmatrix} \Sigma_c & 0_{m \times (n-m)} \end{bmatrix} \begin{bmatrix} V_c^T \\ V_2^T \end{bmatrix} \tag{3}$$

Argue that Σ_c is invertible. What are the matrix elements $\Sigma_{c_{ij}}$ and $\Sigma_{c_{ij}}^{-1}$?

Answer

Since the matrix A is of rank m, the entire diagonal of the matrix Σ_c will consist of nonzero entries. Thus Σ_c is invertible.

The matrix elements of Σ_c are given by:

$$\Sigma_{c_{ij}} = \begin{cases} 0 & i \neq j \\ \sigma_i & i = j \end{cases}$$

For Σ_c^{-1}, they are given by

$$\Sigma_{c_{ij}}^{-1} = \begin{cases} 0 & i \neq j \\ \frac{1}{\sigma_i} & i = j \end{cases}$$

c) Use the SVD of A to show that the expression for the minimum-norm solution from equation 3 can also be written as

$$\vec{x} = A^T(AA^T)^{-1}\vec{y}$$

HINT: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ for invertible matrices $A,\ B,\ C$.

Answer

We begin by substituting the SVD of A into the given expression:

$$\begin{aligned}
\vec{x} = A^T(AA^T)^{-1}\vec{y} &= (U\Sigma V^T)^T(U\Sigma V^T(U\Sigma V^T)^T)^-1\vec{y} \\
&= V\Sigma^T U^T(U\Sigma V^T V\Sigma^T U^T)^{-1}\vec{y} \\
&= V\Sigma^T U^T(U\Sigma\Sigma^T U^T)^{-1}\vec{y} \\
&= V\Sigma^T U^T(U\Sigma_c^2 U^T)^{-1}\vec{y} \\
&= V\Sigma^T U^T(U(\Sigma_c^{-1})^2 U^T)\vec{y} \\
&= V\Sigma^T(\Sigma_c^{-1})^2 U^T\vec{y}
\end{aligned}$$

where in the 5th line we used $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ for invertible matrices $A,\ B,\ C$. Further writing V and Σ^T in block matrix form,

$$\begin{aligned}
\vec{x} = A^T(AA^T)^{-1}\vec{y} &= \begin{bmatrix} V_c & V_2 \end{bmatrix}\begin{bmatrix} S \\ 0 \end{bmatrix}(\Sigma_c^{-1})^2 U^T\vec{y} \\
&= V_c\Sigma_c(\Sigma_c^{-1})^2 U^T\vec{y} \\
&= V_c\Sigma_c^{-1}U^T\vec{y}
\end{aligned}$$

Thus we have shown that the two expressions are equivalent.

3 Minimum Energy Control

Consider the system

$$\vec{x}(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

Our goal is to reach the target state

$$\vec{x}(5) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

starting at $\vec{x}(0) = 0$.

a) Find the input sequence $u(0), u(1), u(2), u(3), u(4)$ that achieves this with the least possible "energy," as defined by

$$E = u(0)^2 + u(1)^2 + u(2)^2 + u(3)^2 + u(4)^2.$$

Find the value of E for the sequence you computed.

Answer

Since $\vec{x}(0) = 0$, we have

$$\vec{x}(5) = \begin{bmatrix} \vec{b} & A\vec{b} & A^2\vec{b} & A^3\vec{b} & A^4\vec{b} \end{bmatrix} \begin{bmatrix} u(4) \\ u(3) \\ u(2) \\ u(1) \\ u(0) \end{bmatrix}$$

Substituting A, $\vec{b}$, and the target value $\vec{x}(5)$:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}}_{C} \begin{bmatrix} u(4) \\ u(3) \\ u(2) \\ u(1) \\ u(0) \end{bmatrix}.$$

Then the minimum-norm solution is

$$\begin{bmatrix} u(4) \\ u(3) \\ u(2) \\ u(1) \\ u(0) \end{bmatrix} = C^T(CC^T)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.2 \\ -0.1 \\ 0 \\ 0.1 \\ 0.2 \end{bmatrix}$$

and the energy is

$$(-0.2)^2 + (-0.1)^2 + 0 + (0.1)^2 + (0.2)^2 = 0.1.$$

b) Show that, if we reach the target state $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ at $t = 2$ and apply a zero input henceforth ($u(2) = u(3) = u(4) = \cdots = 0$) then

$$\vec{x}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad t = 2, 3, 4, 5, \ldots$$

Answer

If we substitute $u(t) = 0$ and

$$\vec{x}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

in

$$\vec{x}(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

we get

$$\vec{x}(t+1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Thus, once we reach the target state we can stay there by applying zero inputs afterwards.

c) With the result of part (b) in mind, find inputs $u(0), u(1)$ such that

$$\vec{x}(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and apply $u(2) = u(3) = u(4) = 0$ so

$$\vec{x}(5) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

as well. Find the energy E for the resulting input sequence and compare it to the one in part (a).

Answer

To reach the target step in two steps we need to solve

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{x}(2) = \begin{bmatrix} \vec{b} & A\vec{b} \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix},$$

that is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The energy is

$$(1)^2 + (-1)^2 = 2.$$

4 Uncontrollability

Consider the following discrete-time system with the given initial state:

$$\vec{x}(t+1) = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t)$$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

a) Is the system controllable?

Answer

$$C = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Since the controllability matrix C only has rank 2, the system is not controllable.

b) Is it possible to reach $\vec{x}(T) = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ for some $t = T$? For what input sequence $u(t)$ up to $t = T - 1$?

Answer

$$\vec{x}(1) = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(0) = \begin{bmatrix} 2 \\ -3 \\ 2u(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$$

Yes, we can actually reach $\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$ within one timestep by setting $u(0) = -1$.

c) Find the set of all possible states reachable after two timesteps.

Answer

$$\vec{x}(1) = \begin{bmatrix} 2 \\ -3 \\ 2u(0) \end{bmatrix}$$

$$\vec{x}(2) = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 2u(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(1) = \begin{bmatrix} 4 \\ -6 + 2u(0) \\ -3 + 2u(1) \end{bmatrix}$$

Since we can set $u(0)$ and $u(1)$ arbitrarily, we can reach any state of the form $\begin{bmatrix} 4 \\ c_1 \\ c_2 \end{bmatrix}$ after two timesteps.

d) Is it possible to reach $\vec{x}(T) = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some $t = T$? For what input sequence $u(t)$ up to $t = T - 1$?

Answer

No, we notice that $x_1(t) = 2x_1(t-1)$, so $x_1(t) = 2^t$. Since $x_1(t)$ will continue to grow exponentially, $x_1(t) \neq -2$ for all t. Therefore, we will never be able to reach $\vec{x}(T) = \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$ for some $t = T$. This counterexample shows why our system is not controllable.