1 Controllability

We are given a discrete-time state space system, where $\vec{x}$ is our state vector, A is the state space model, B is the input matrix, and $\vec{u}$ is the control input.

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$

We want to know if this system is "controllable"; if given set of inputs, we can reach any state in our state-space after a finite number time steps. This has an important physical meaning; if a system is controllable, we can travel anywhere in the system it is living in given enough control inputs.

Constructing the Controllability Matrix

To figure out if a system is controllable, we start by looking at a simplified problem. Suppose the initial state $\vec{x}[0] = \vec{0}$ and we would like to reach a certain state $\vec{x}_0$. If a system were controllable, we can give a set of control inputs $\vec{u}[t]$ to reach this state $\vec{x}_0$. Let's start by analyzing what happens after one time step.

$$\vec{x}[1] = A\vec{x}[0] + B\vec{u}[0] = A\vec{0} + B\vec{u}[0] = B\vec{u}[0] \tag{1}$$

This shows us that we can go anywhere spanned by B in the first time step. Now let us push the system another time step.

$$\begin{align} \vec{x}[2] &= A\vec{x}[1] + B\vec{u}[1] = A(A\vec{x}[0] + B\vec{u}[0]) + B\vec{u}[1] \tag{2} \\ &= AB\vec{u}[0] + B\vec{u}[1] \tag{3} \end{align}$$

Similarly, at this time step, we can go anywhere spanned by $\begin{bmatrix} B & AB \end{bmatrix}$. Every time step adds another degree of freedom to the system.

To generalize this relation, after n time steps, we get the following:

$$\begin{align} \vec{x}[n] &= A\vec{x}[n-1] + B\vec{u}[n-1] \tag{4} \\ &= A^{n-1}B\vec{u}[0] + A^{n-2}B\vec{u}[1] + \cdots + AB\vec{u}[n-2] + B\vec{u}[n-1] \tag{5} \end{align}$$

This shows that after n time steps, we can go anywhere spanned by the columns of the matrix C defined below. This is called the "controllability" matrix.

$$C = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-2}B & A^{n-1}B \end{bmatrix} \tag{6}$$

If this matrix is of rank n (the dimension of our state space), then our system is controllable. But what if these aren't enough steps and the system can be controlled only in $n+1$ steps? What is the maximal number of steps we need to take to have a long sequence of control inputs that $\{B, AB, A^2B, \ldots\}$ spans the state space?

Cayley-Hamilton Theorem

These questions are answered by the Cayley-Hamilton theorem. The Cayley-Hamilton theorem says that higher order powers of A can be expressed as a linear combination of lower order matrix powers of A. Specifically if A is an $n \times n$, matrix, the highest order unique power of A is A^{n-1}. Thus, if we keep applying control inputs past n time steps, our control inputs will be a linear combination of the previous control inputs and cannot increase the rank of the controllability matrix.

Definition of Controllability

This also works for continuous time systems; instead of incrementing the time steps in our system by 1 every time, we increment by Δt, $2\Delta t$, etc. The math and our controllability test work out to be exactly the same! Putting all of this together, we get the following:

$$\dot{\vec{x}}(t) = A\vec{x}(t) + B\vec{u}(t) \quad \text{or} \quad \vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$
$$C = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-2}B & A^{n-1}B \end{bmatrix}$$

Given a continuous or discrete time system $\vec{x}$ of dimension n, the system is controllable if its controllability matrix C is of rank n. If a system is controllable, then given any starting position $\vec{x}[0]$, it takes a maximum of n control inputs over n time steps for the system to reach any final state $\vec{x}_0$.

2 Deadbeat Control

Consider the system

$$\vec{x}[t+1] = A\vec{x}[t] + Bu[t] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t].$$

a) Is this system controllable?

Answer

We compute the controllability matrix C:

$$C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

This matrix has a rank of 2, so the system is controllable.

b) For which initial states $\vec{x}[0]$ is there a control that will bring the state to zero in a single time step?

Answer

To find the initial states that can be brought to zero in a single step, we solve:

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[0] \\ &= \begin{bmatrix} x_1[0] - x_2[0] \\ x_2[0] - x_1[0] + u[0] \end{bmatrix} \\ \implies \; 0 &= x_1[0] - x_2[0]. \end{aligned}$$

Therefore, there is a one-dimensional subspace $\{x_1[0] - x_2[0] = 0\}$ of initial states that can be brought to zero in one step.

c) For which initial states $\vec{x}[0]$ is there a control that will bring the state to zero in two time steps?

Answer

To find the initial states that can be brought to zero in two steps, we solve:

$$\begin{aligned}\begin{bmatrix}0\\0\end{bmatrix} &= \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}x_1[1]\\x_2[1]\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u[1]\\ &= \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}x_1[0]-x_2[0]\\x_2[0]-x_1[0]+u[0]\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u[1]\\ &= \begin{bmatrix}2x_1[0]-2x_2[0]-u[0]\\2x_2[0]-2x_1[0]+u[0]+u[1]\end{bmatrix}\end{aligned}$$

Therefore, any initial state can be brought to zero in two steps using an appropriate choice of inputs $u[0]$ and $u[1]$.

d) Now let $\vec{x}[0] = \begin{bmatrix}1\\0\end{bmatrix}$ be the initial state. Give a set of control inputs $u[0]$ and $u[1]$ to bring to system to $\vec{y} = \begin{bmatrix}0\\1\end{bmatrix}$ in two time steps.

Answer

Expanding out the difference equation, we see that after two time steps

$$\vec{x}[2] = A^2\vec{x}[0] + ABu[0] + Bu[1] \tag{7}$$

This means that we can solve for the control inputs $u[0]$ and $u[1]$ through the following system of equations

$$C\begin{bmatrix}u[1]\\u[0]\end{bmatrix} = \vec{x}[2] - A^2\vec{x}[0] \implies \begin{bmatrix}u[1]\\u[0]\end{bmatrix} = C^{-1}(\vec{x}[2] - A^2\vec{x}[0]) \tag{8}$$

Solving the system yields the following pair of inputs

$$u[0] = 2, u[1] = 1 \tag{9}$$

3 Discretization

Consider a cart of mass M, pushed with a force $u(t)$ with position, $x(t)$, and velocity, $v(t)$. Hence, we have:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = v(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = \frac{u(t)}{M}$$

We will apply a constant input between any time $t \in [nT, nT + T)$. Here T is our time between samples.

a) Find a discretized system of equations for this system.

Answer

To discretize this, we first solve the differential equation for $v(t)$ for interval $t \in [nT, nT + T)$

$$\int_{nT}^{t} dv = \int_{nT}^{t} \frac{u(nT)}{M}\,\mathrm{d}\tau$$
$$v(t) - v(nT) = (t - nT)\frac{u(nT)}{M}$$

Now we substitute the expression for $v(t)$ into the first differential equation and solve for $x(t)$

$$\int_{nT}^{t} dx = \int_{nT}^{t} \left(v(nT) + (\tau - nT)\frac{u(nT)}{M}\right)\mathrm{d}\tau$$

Change the variable of integration to $s = \tau - nT$:

$$\begin{aligned} x(t) - x(nT) &= \int_{0}^{t-nT} \left(v(nT) + s\frac{u(nT)}{M}\right)\mathrm{d}s \\ &= v(nT)(t - nT) + \frac{u(nT)}{2M}(t - nT)^2 \end{aligned}$$

Lastly, we evaluate both states time $t = nT + T$ to get the discretized model

$$x(nT + T) = x(nT) + Tv(nT) + \frac{T^2}{2}\frac{u(t)}{M}$$
$$v(nT + T) = v(nT) + T\frac{u(nT)}{M}$$
$$\begin{bmatrix} x[n+1] \\ v[n+1] \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x[n] \\ v[n] \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2M} \\ \frac{T}{M} \end{bmatrix} u[n]$$

b) Is the discretized system controllable?

Answer

The controllability matrix

$$C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} \frac{T^2}{2M} & \frac{3T^2}{2M} \\ \frac{T}{M} & \frac{T}{M} \end{bmatrix}$$

has nonzero determinant $\det(C) = -\frac{T^3}{M^2}$ since $T, M > 0$. Therefore, the controllability matrix has rank 2 and the discretized system is controllable.