1 Continuous and discrete time

There are two different dialects for modeling change over time. Thus far we have modeled real-life events using differential equations and initial conditions. For example, the voltage across a capacitor connected to a voltage source by a resistor is fully described by the following differential equation and initial conditions.

$$\frac{\mathrm{d}}{\mathrm{d}t} v_C(t) = -\frac{1}{RC} v_C(t) + \frac{1}{RC} v_{\text{in}}(t), \quad v_C(0) = v_0 \tag{1}$$

Abstracting away particulars, *continuous-time* scalar linear systems can be represented in variants of the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda x(t) + \mu u(t), \quad x(0) = x_0. \tag{2}$$

This discussion will introduce *discrete-time* scalar linear systems, which have models similar to the following:

$$x[t+1] = a x[t] + b u[t], \quad x[0] = x_0. \tag{3}$$

Notice that evolution is represented by defining the transition from $x[t]$ to $x[t+1]$. The state x is not a continuous function of time, but a sequence of individual moments. Can you think of systems in life that are naturally more susceptible to discrete-time modeling?

2 Dirty Dishes

I am a trip planner who lodges travellers at Bob's Bed and Breakfast. At the beginning of each day, Bob will do half of the dirty dishes in the sink. During the day, each of his guests will use 4 pounds of dishes minus an eighth pound of dishes for each pound of dishes already in the sink at the beginning of the day (as Bob's kitchen gets too messy).

a) What is the state vector for Bob's kitchen sink system? What are the inputs? Write out the state space model.

Answer

The dishes in the sink are the state variable x. The number of guests are the input u.

$$x[t+1] = \frac{1}{2} x[t] + \left(4 - \frac{1}{8} x[t]\right) u[t]$$

b) Explain why Bob's kitchen is not a linear system.

Answer

It is not a linear system since the state variable is multiplied by the input.

c) On Wednesday morning (before Bob gets up), there are 4 pounds of dishes in the sink. On Wednesday, Bob has 4 guests, and on Thursday, he has 5 guests. How many pounds of dishes are in the sink after Thursday?

Answer

$$x[1] = \frac{1}{2}(4) + \left(4 - \frac{1}{8}(4)\right)(4) = 16$$
$$x[2] = \frac{1}{2}(16) + \left(4 - \frac{1}{8}(16)\right)(5) = 18$$

d) I am a very eccentric trip planner and I want Bob to have exactly 12 pounds of dishes in his sink. He has 24 pounds of dishes in his sink. How many guests should I lodge at Bob's Bed and Breakfast today? How many guests should I lodge tomorrow?

Answer

$$12 = \frac{1}{2}(24) + \left(4 - \frac{1}{8}(24)\right)u[0]$$
$$u[0] = 0$$

$$12 = \frac{1}{2}(12) + \left(4 - \frac{1}{8}(12)\right)u[1]$$
$$u[1] = \frac{12}{5}$$

e) Now suppose 5 guests come to Bob's kitchen every day. At the equilibrium state, how many pounds of dishes will remain in the sink?

Answer

Given $u^* = 5$, let x^* be the equilibrium state. Then we know that

$$x^* = \frac{1}{2}x^* + (4 - \frac{1}{8}x^*) \cdot u^* \quad (4)$$
$$= \frac{1}{2}x^* + (4 - \frac{1}{8}x^*) \cdot 5 \quad (5)$$

Solving for x^* yields $x^* = \frac{160}{9}$.

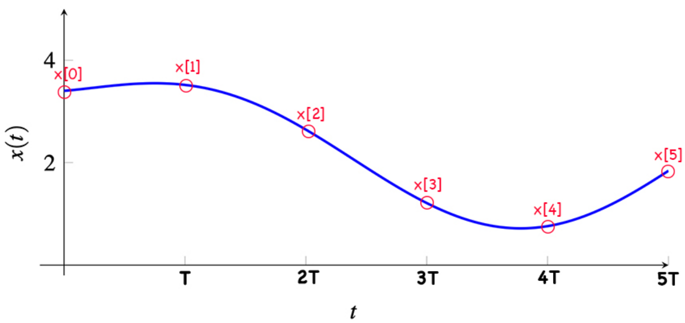
3 Differential equations with piecewise constant inputs

Let $x(\cdot)$ be a solution to the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \lambda \left(x(t) - u(t) \right). \tag{6}$$

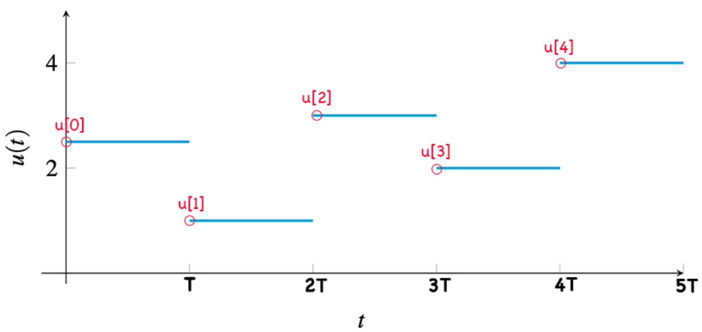
Let $T > 0$. Let $x[\cdot]$ "sample" $x(\cdot)$ as follows:

$$x[n] = x(nT). \tag{7}$$



Assume that $u(\cdot)$ is constant between samples of $x(\cdot)$, i.e.

$$u(t) = u[n] \quad \text{when} \quad nT \leq t < (n+1)T. \tag{8}$$



a) We will approach solving this differential equation iteratively in intervals of size T. What is the solution $x(t)$ for $t \in [0, T)$?

Answer

In the interval $[0, T)$, the input $u(t)$ will be a constant equal to $u[0]$. Therefore, the differential equation can be written as

$$\frac{d}{dt} x(t) = \lambda(x(t) - u[0]) \tag{9}$$

Recall that the solution to this differential equation is

$$x(t) = x(0)e^{\lambda t} + (1 - e^{\lambda t})u[0] \tag{10}$$

b) Using your solution from the previous part, sample it at $t = T$ to write $x[1]$ in terms of $x[0]$ and $u[0]$.

Answer

Plugging in $t = T$, we see that

$$x[1] = x(T) = x[0]e^{\lambda T} + (1 - e^{\lambda T})u[0] \tag{11}$$

c) Now for a general time-step n, write $x[n+1]$ in terms of $x[n]$ and $u[n]$. Conclude that the sampled system of a continuous-time linear system is in fact a discrete-time linear system.

Answer

As $u = u[n]$ is constant between samples $x[n]$ and $x[n+1]$, the following differential equation and initial conditions describe what is happening to x during this interval:

$$\frac{d}{dt}x(t) = \lambda(x(t) - u[n]), \quad x(nT) = x[n]. \tag{12}$$

This differential equation can be solved by guessing the solution $x(t) = K_1 e^{\lambda t} + K_2$

$$\frac{d}{dt}x(t) = K_1 \lambda e^{\lambda t} = \lambda(x(t) - u[n]) \tag{13}$$

$$= \lambda(K_1 e^{\lambda t} + K_2 - u[n]) \tag{14}$$

Thus $K_2 = u[n]$ and we plug in the initial condition $x(nT)$ to solve for K_1

$$x(nT) = K_1 e^{\lambda n T} + u[n] \tag{15}$$

Solving yields $K_1 = e^{-\lambda n T}(x(nT) - u[n])$.

Now that we have a solution to the differential equation, we can sample it at $t = (n+1)T$

$$x[n+1] = x((n+1)T) = e^{-\lambda n T}(x(nT) - u[n])e^{\lambda(n+1)T} + u[n] \tag{16}$$

Simplifying the expression, it follows that

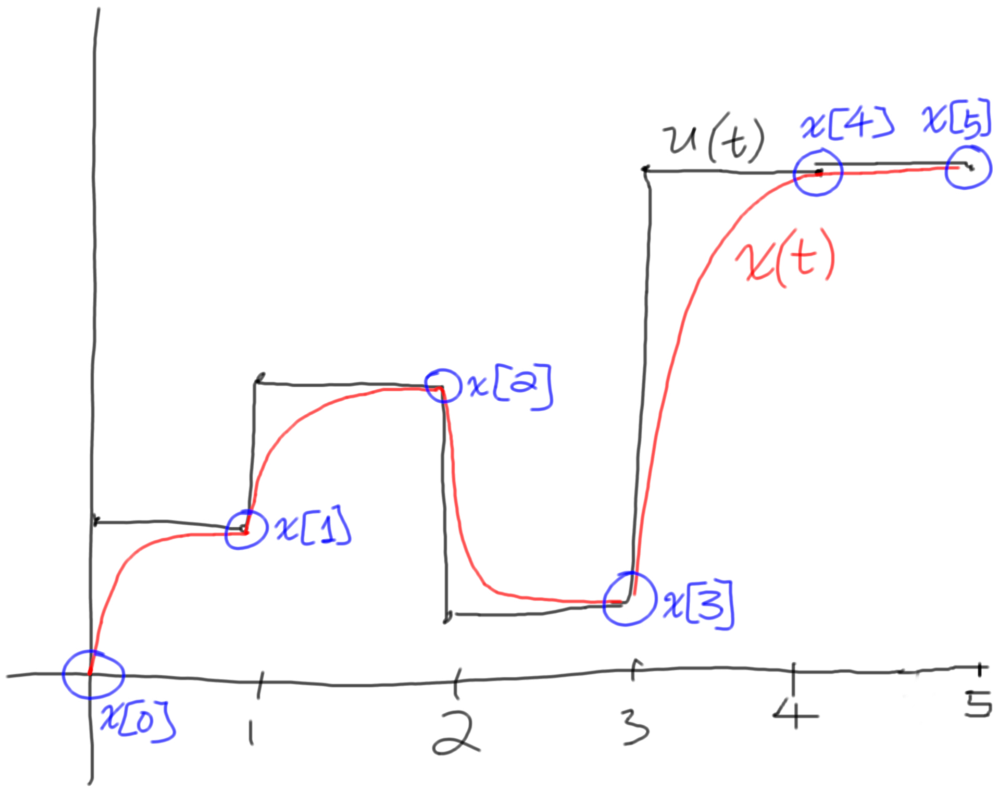
$$x[n+1] = e^{\lambda T}x[n] + (1 - e^{\lambda T})u[n] \tag{17}$$

Thus the sampled system is a linear discrete-time system $x[n+1] = ax[n] + bu[n]$ with $a = e^{\lambda T}$ and $b = 1 - e^{\lambda T}$.

d) Let $T = 1$ and $\lambda = -100$. Sketch a piecewise constant input $u[\cdot]$ of your choice, then sketch $x(t)$. Mark $x[n]$. Your sketch doesn't have to be exact, but you should be able to supply analysis to justify why it looks a certain way: how are you using the fact that λT is large and negative?

Answer

A typical drawing might look similar to this:



Notice that the displacement between $x(t)$ and its moving target $u(t)$ is always in exponential decay (it is proportional to $z(t)$). Because λT is large and negative, $e^{\lambda T} \approx 0$, so

$$x[n+1] = e^{\lambda T} x[n] + \left(1 - e^{\lambda T}\right) u[n] \tag{18}$$
$$\approx u[n] \tag{19}$$

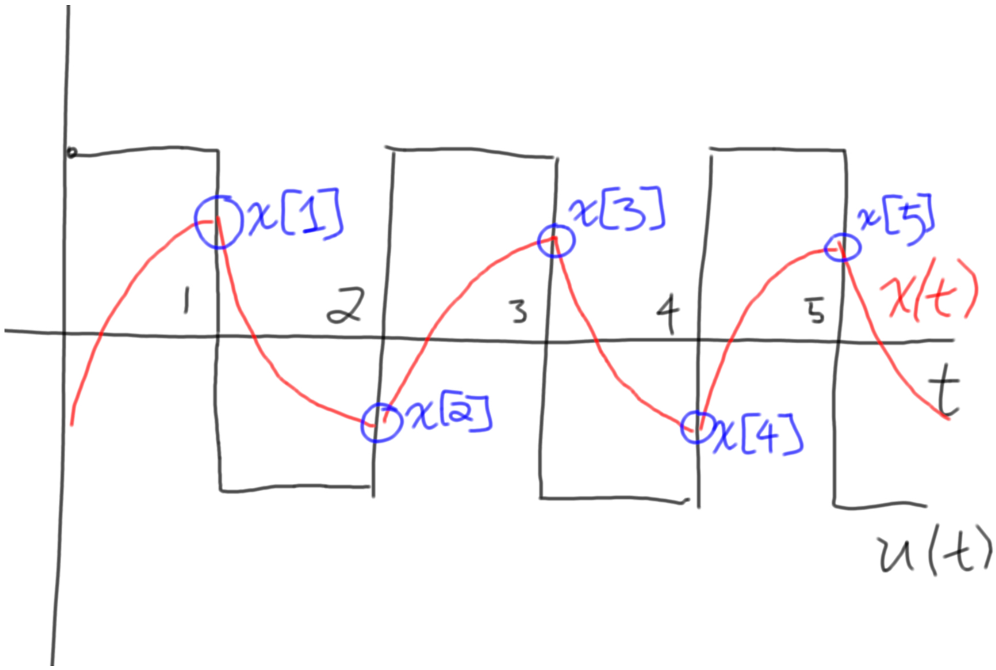
e) Let $T = 1$ and $\lambda = -1$. Define $u[n]$ as follows:

$$u[n] = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases}. \tag{20}$$

Sketch $x(t)$.

Answer

Notice how u, which is x's target, is flipping so quickly that x never gets close to the finish line. It gets partway there and then is told to turn around. An approximate sketch (with features exaggerated) would look like this:


x[1]
x[2]
x[3]
x[4]
x[5]
1
2
3
4
5
x(t)
t
u(t)