1 Single-dimensional linearization

This is an exercise in linearizing a scalar system. The scalar nonlinear differential equation we have is

$$\frac{d}{dt}x(t) = \sin\bigl(x(t)\bigr) + u(t). \tag{1}$$

a) Find the equilibrium points for $u^* = 0$. You can do this by sketching $\sin(x)$ for $-4\pi \le x \le 4\pi$ and intersecting it with the horizontal line at 0. This will give you the equilibrium points x^* where $sin(x^*) + u^* = 0$.

Answer

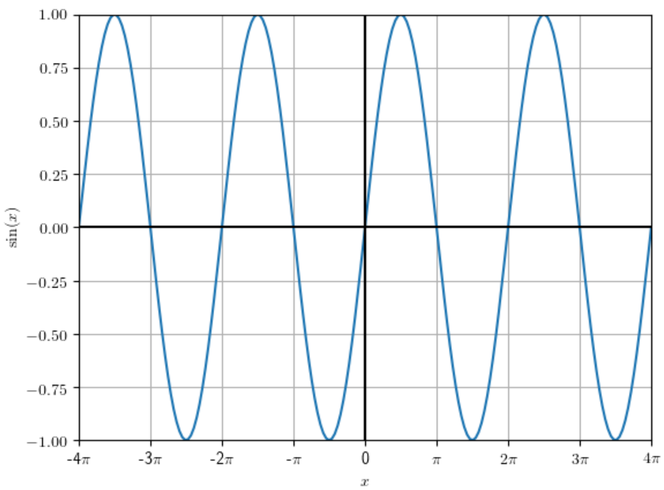


Figure 1: Plot of $\sin(x)$

We can see that all the multiples of π are where the line intersects the sine wave. This means that $x_m^* = m\pi$ is an equilibrium poin for any integer m.

b) Linearize the system (1) around the equilibrium $(x_0^*, u^*) = (0, 0)$. **What is the resulting linearized scalar differential equation for $\tilde{x}(t) = x(t) - x_0^* = x(t) - 0$, involving $\tilde{u}(t) = u(t) - u^* = u(t) - 0$?**

Answer

First substituting for x and u in (1), then substituting the linearization for $\sin(x)$, we get,

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \sin\bigl(x(t)\bigr) + u(t) \\
\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t} &= \sin\bigl(\tilde{x}(t)\bigr) + \tilde{u}(t) \\
&\approx \sin\bigl(x_0^*\bigr) + \tilde{x}(t)\frac{d}{dx}\sin(x)\Big|_{x=x_0^*} + \tilde{u}(t) \\
&= 0 + \tilde{x}(t)\cos(0) + \tilde{u}(t) \\
&= \tilde{x}(t) + \tilde{u}(t)
\end{aligned}$$

Notice that this approximate equality has been made an exact equality by calling the approximation error a disturbance $w(t)$.

2 Jacobian Warm-Up

Consider the following function $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$

$$f(x_1, x_2) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ f_3(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_1^2 - x_2^2 \\ x_1^2 + x_1 x_2^2 \\ x_1 \end{bmatrix}$$

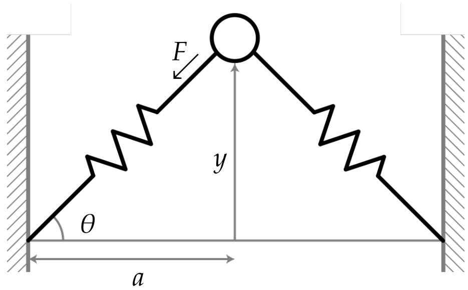
Calculate its Jacobian.

Answer

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}\vec{x}} &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 & -2x_2 \\ 2x_1 + x_2^2 & 2x_1 x_2 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

3 Linearization

Consider a mass attached to two springs:



We assume that each spring is linear with spring constant k and resting length X_0. We want to build a state space model that describes how the displacement y of the mass from the spring base evolves. The differential equation modeling this system is $\frac{d^2y}{dt^2} = -\frac{2k}{m}(y - X_0\frac{y}{\sqrt{y^2+a^2}})$.

a) Write this model in state space form $\dot{x} = f(x)$.

Answer

We introduce states $x_1 = y$ and $x_2 = \dot{y}$. Writing the model in state space form gives

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{-2k}{m}\left(x_1 - X_0\dfrac{x_1}{\sqrt{x_1^2+a^2}}\right)\end{bmatrix}.$$

b) Find the equilibrium of the state-space model. You can assume $X_0 < a$.

Answer

We find the equilibrium by solving $0 = \dot{x} = f(x)$:

$$\begin{bmatrix}0\\ 0\end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{-2k}{m}\left(x_1 - X_0\dfrac{x_1}{\sqrt{x_1^2+a^2}}\right)\end{bmatrix}.$$

The unique solution is the equilibrium at $(x_1, x_2) = (0, 0)$.

c) Linearize your model about the equilibrium.

Answer

$$\left.\begin{bmatrix}\dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2}\\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2}\end{bmatrix}\right|_{x=(0,0)} = \left.\begin{bmatrix} 0 & 1 \\ -\dfrac{2k}{m}\left(1 - X_0\dfrac{a^2}{(x_1^2+a^2)^{3/2}}\right) & 0\end{bmatrix}\right|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ -\dfrac{2k}{m}\left(1 - \dfrac{X_0}{a}\right) & 0\end{bmatrix}$$

So the linearized system is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\dfrac{2k}{m}\left(1 - \dfrac{X_0}{a}\right) & 0\end{bmatrix} x.$$